LESSONS

INTRODUCTORY TO THE

MODERN HIGHER ALGEBRA.

BY

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TO

A. CAYLEY, ESQ., AND J. J. SYLVESTER, ESQ.,

I BEG TO INSCRIBE

THIS ATTEMPT TO RENDER SOME OF THEIR DISCOVERIES BETTER KNOWN,

In Acknowledgment

OF

THE OBLIGATIONS I AM UNDER, NOT ONLY TO THEIR PUBLISHED WRITINGS,

BUT ALSO

TO THEIR INSTRUCTIVE CORRESPONDENCE.

PREFACE.

The following pages have grown out of what was originally intended to be an Appendix to a work on the Geometry of Three Dimensions, which I have had for some time in preparation. I found that it would be desirable to add at the end an explanation of such algebraical principles or terms as I might find it convenient to employ, and for an account of which I could not refer to any elementary works already in the hands of students. I had added a similar Appendix to a "Treatise on the Higher Plane Curves," which I published a few years ago, but this branch of Mathematics has made such rapid progress of recent years that my former sketch has become now out of date, and would not suffice for my present purpose. In attempting to re-write it, I have been led on by degrees to enlarge my plan, and indeed at one time contemplated the publication of an extended course of Higher Algebra, including, for example, in addition to the topics touched on in the following pages, those treated of in M. Serret's Lessons, a work from which I have borrowed the title of mine, though in other respects his Lessons and mine have little in common. Ultimately I have limited these Lessons to what I have called the *Modern* Higher Algebra, but which I might with greater precision have

called the Algebra of Linear Transformations, a department of Analysis which has been created during the last twenty years.

It may be said to date from a paper published by Professor Boole in the "Cambridge Mathematical Journal" for November, 1841, which was mainly occupied with applications of the following theorem: Let an ordinary algebraic equation be made homogeneous by writing $x : y$ for x; let these variables then be linearly transformed by writing $\lambda x + \mu y$, $\lambda' x + \mu' y$ for x and y; and let $V = 0$ be the condition that the transformed equation shall have equal roots; then the function V will be equal to the similar function for the original equation multiplied by a function of λ, μ, λ', μ', and not involving the coefficients of the original equation. The truth of this principle is manifest from the fact that if the original equation have a square factor, the transformed must have one likewise; that, therefore, one of the functions in question must vanish whenever the other does, and, consequently, the one must contain the other as a factor; and since each of the coefficients of the transformed only contains in the first degree the coefficients of the original, the two functions, being of the same degree in those coefficients, can only differ by a quantity independent of them. Similar reasoning applies to functions of any number of variables.

Mr. Boole having in the paper just cited made important use of the principle here enunciated, Mr. Cayley subsequently proposed to himself the problem to determine *à priori* what functions of the coefficients of a given equation possess this property of *invariance:* viz., that when the equation is linearly transformed, the same function of the new coefficients

shall be equal to the given function, multiplied by a quantity independent of these coefficients. The result of Mr. Cayley's investigations was, to discover that the property of invariance was not peculiar to the functions which had been discussed by Mr. Boole, and to bring to light other important functions possessing the same property. Subsequently it was found that functions could be formed involving the variables as well as the coefficients, and possessing the same permanent relation to the original equation: that it was thus possible to form two functions connected by a certain relation between their coefficients, and such that when both functions were linearly transformed, the same relation should continue to exist between their coefficients. Functions so related have been called covariants.

The geometrical importance of this theory is now manifest. When we are given the equation of any curve or surface, the theory of linear transformations at once presents us with equations representing other curves and surfaces, and possessing permanent relations to the given one, which will be unaffected by any change of the axes of co-ordinates. And in like manner the same theory presents us with certain functions of the coefficients of the given equation, the vanishing of which must express a property of the given curve or surface, wholly independent of the choice of axes. Besides these geometrical applications, the theory has other important uses, which I shall not stop to enumerate.

That so valuable a theory is not yet as well known in this country as it deserves to be, must arise from the difficulty of becoming acquainted with it. I am sure that there are many mathematicians who find with regret that the more recent memoirs on this subject are unin-

telligible to them, in consequence of their having overlooked the earlier memoirs, of the importance of which they were not aware at the time that they were published. And I feel that such persons will be ready to welcome an elementary guide to this branch of Algebra. I am very far from being satisfied with the manner in which I have here supplied this want. There are some parts of the following Lessons which I believe might have been compressed with advantage, and the space so gained might have been filled up with more important matter. I might plead as my apology the little leisure that other engagements have left me either for writing or for the necessary reading; if it were not that I have a better apology in the fact that no one more competent has fulfilled the task which is here attempted.

A considerable part of the following pages having been written at a distance from books, I have been able to add fewer references than I could have wished to my sources of information. It is, therefore, right to state here that these Lessons make no pretension to originality, and that far the greater part of them is derived from the Papers of Mr. Cayley, and of his fellow-labourer, Mr. Sylvester, in the "Cambridge and Dublin Mathematical Journal," and in the "Philosophical Transactions." The only other important acknowledgments which it occurs to me to add to those made in the text are, that in Lesson VIII., in which the subject of Invariants is introduced, I have mainly followed the method of Professor Boole's Memoir ("Cambridge and Dublin Mathematical Journal," vol. vi., p. 87), a paper to which I look back with interest, as that from which I myself derived my first clear ideas as to the nature and objects of the Theory of Linear Transformations.

On the subject of the first three Lessons, I have to refer to Mr. Spottiswoode's "Theorems concerning Determinants," contributed to "Crelle's Journal," and also published separately by Longman and Co.; and to Brioschi's "Theory of Determinants," translated from Italian into French by M. Combescure. On comparing the number of pages occupied by my Lessons on Determinants with those devoted to the subject by M. Brioschi or by Mr. Spottiswoode, it will be evident that I have made many omissions, my object being principally to explain such parts of the theory as I should afterwards have occasion to employ. The reader will find in Mr. Spottiswoode's Preface a historical sketch of the progress of this theory.

ERRATUM.

Note, p. 21, line 8. The values of a_0, a_1; a_m, a_{m-1} are *transposed.*

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LESSON I.

DETERMINANTS.—PRELIMINARY ILLUSTRATIONS.

1. If we are given n homogeneous equations of the first degree between n variables, we can eliminate the variables, and obtain a result involving the coefficients only, which is called the *determinant* of those equations. We shall, in what follows, give rules for the formation of these determinants, and shall state some of their principal properties; but we think that the general theory will be better understood if we first give illustrations of its application to the simplest examples.

Let us commence, then, with two equations between two variables—

$$a_1x + b_1y = 0, \qquad a_2x + b_2y = 0.$$

The variables are eliminated by adding the first equation multiplied by b_2 to the second multiplied by $-b_1$, when we get $a_1b_2 - a_2b_1 = 0$, which is the determinant required. The ordinary notation for this determinant is

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}.$$

We shall, however, often, for brevity, write (a_1b_2) to express this determinant, leaving the reader to supply the term with the negative sign; and in this notation it is obvious that $(a_1b_2) = -(a_2b_1)$. The coefficients a_1, b_1, &c., which enter into the expression of a determinant, are called the *constituents* of that determinant, and the products a_1b_2, &c., are called the *elements* of the determinant.

2. It can be verified at once that we should have obtained the same result if we had eliminated the variables between the equations

$$a_1x + a_2y = 0, \qquad b_1x + b_2y = 0.$$

In other words—

$$\begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} = \begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix}$$

or the value of the determinant is not altered if we write the horizontal rows vertically, and *vice versâ*.

3. The notation

$$\left\| \begin{matrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \end{matrix} \right\|$$

(where the number of columns is greater than the number of rows) is used to denote the three determinants, which can be obtained by suppressing in turn each one of the columns, viz.: the three determinants, (a_2b_3), (a_3b_1), (a_1b_2). These three determinants are connected by the following identical relations:

$$a_1(a_2b_3) + a_2(a_3b_1) + a_3(a_1b_2) = 0,$$
$$b_1(a_2b_3) + b_2(a_3b_1) + b_3(a_1b_2) = 0.$$

These relations can be verified at once by writing them at full length; for instance,

$$a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0.$$

But the following method will serve better to show how to obtain corresponding identical equations in the case of determinants of higher order. Take the two equations,

$$a_1x + a_2y + a_3z = 0, \qquad b_1x + b_2y + b_3z = 0;$$

and by eliminating y and x alternately, solve for x and y in terms of z. We shall find

$$(a_1b_2)x = (a_2b_3)z, \qquad (a_1b_2)y = (a_3b_1)z,$$

which values, substituted in the two original equations, give the identities already written.

4. Let us now proceed to a system of three equations—

$$a_1x + b_1y + c_1z = 0, \qquad a_2x + b_2y + c_2z = 0, \qquad a_3x + b_3y + c_3z = 0.$$

Then, if we multiply the first by (a_2b_3), the second by (a_3b_1), the third by (a_1b_2), and add; the coefficients of x and y will vanish in virtue of the identical relations of Art. 3, and the determinant required is

$$c_1(a_2b_3) + c_2(a_3b_1) + c_3(a_1b_2) = 0;$$

or, writing at full length,

$$c_1a_2b_3 - c_1a_3b_2 + c_2a_3b_1 - c_2a_1b_3 + c_3a_1b_2 - c_3a_2b_1 = 0.$$

It may also be written in either of the forms,

$$a_1(b_2c_3) + a_2(b_3c_1) + a_3(b_1c_2) = 0, \quad b_1(c_2a_3) + b_2(c_3a_1) + b_3(c_1a_2) = 0.$$

This determinant is expressed by the notation

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

though we shall often use for it the abbreviation $(a_1b_2c_3)$.

It is useful to observe that

$$(a_2b_3c_1) = (a_1b_2c_3), \text{ but } (a_1b_3c_2) = -(a_1b_2c_3).$$

For by analogy of notation,

$(a_2b_3c_1) = a_2(b_3c_1) + a_3(b_1c_2) + a_1(b_2c_3)$, which is the same as $(a_1b_2c_3)$, while

$(a_1b_3c_2) = a_1(b_3c_2) + a_3(b_2c_1) + a_2(b_1c_3)$, which is the same as $-(a_1b_2c_3)$.

5. We should have obtained the same result of elimination if we had eliminated between the three equations—

$$a_1x + a_2y + a_3z = 0, \quad b_1x + b_2y + b_3z = 0, \quad c_1x + c_2y + c_3z = 0.$$

For if we proceed on the same system as before, multiplying the first equation by (b_2c_3), the second by (c_2a_3), and the third by (a_2b_3), and add, then the coefficients of y and z vanish, and the determinant is obtained in the form

$$a_1(b_2c_3) + b_1(c_2a_3) + c_1(a_2b_3),$$

which, expanded, is found to be identical with $(a_1b_2c_3)$. Hence

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}$$

or the determinant is not altered by writing the horizontal rows vertically, and *vice versâ;* a property which will be proved to be true of every determinant.

6. Using the notation

$$\left\|\begin{matrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \end{matrix}\right\|$$

to denote the system of determinants obtained by omitting in turn each one of the columns; these four determinants are connected by the relations

$$\begin{aligned} a_1(a_2b_3c_4) - a_2(a_3b_4c_1) + a_3(a_4b_1c_2) - a_4(a_1b_2c_3) &= 0, \\ b_1(a_2b_3c_4) - b_2(a_3b_4c_1) + b_3(a_4b_1c_2) - b_4(a_1b_2c_3) &= 0, \\ c_1(a_2b_3c_4) - c_2(a_3b_4c_1) + c_3(a_4b_1c_2) - c_4(a_1b_2c_3) &= 0. \end{aligned}$$

These relations may be either verified by actual expansion of the determinants, or else may be proved by a method analogous to that used in Art. 3. Take the three equations:

$$\begin{aligned} a_1x + a_2y + a_3z + a_4w &= 0, \\ b_1x + b_2y + b_3z + b_4w &= 0, \\ c_1x + c_2y + c_3z + c_4w &= 0. \end{aligned}$$

Then (as in Art. 5) we can eliminate y and z by multiplying the equations by (b_2c_3), (c_2a_3), (a_2b_3), respectively, and adding, when we get

$$(a_1b_2c_3)\,x + (a_4b_2c_3)\,w = 0.$$

In like manner, multiplying by (b_3c_1), (c_3a_1), (a_3b_1) respectively, we get

$$(a_2b_3c_1)\,y + (a_4b_3c_1)\,w = 0.$$

And in like manner,

$$(a_3b_1c_2)\,z + (a_4b_1c_2)\,w = 0.$$

Now, attending to the remarks about signs (Art. 4), these equations are equivalent to

$$(a_1b_2c_3)x = -(a_2b_3c_4)w,\ (a_1b_2c_3)y = (a_3b_4c_1)w,\ (a_1b_2c_3)z = -(a_4b_1c_2)w.$$

And substituting in the original equations the values for x, y, z, just found, we obtain the identities which it is required to prove.

7. If now we have to eliminate between the four equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1w &= 0, \\ a_2x + b_2y + c_2z + d_2w &= 0, \\ a_3x + b_3y + c_3z + d_3w &= 0, \\ a_4x + b_4y + c_4z + d_4w &= 0, \end{aligned}$$

we have only to multiply the first by $(a_2b_3c_4)$, the second by $-(a_3b_4c_1)$, the third by $(a_4b_1c_2)$, the fourth by $-(a_1b_2c_3)$, and add, when the coefficients of x, y, z vanish identically, and the determinant is found to be

$$d_1(a_2b_3c_4) - d_2(a_3b_4c_1) + d_3(a_4b_1c_2) - d_4(a_1b_2c_3) = 0;$$

or, writing it full length,

$$\begin{aligned} a_1b_2c_3d_4 &- a_1b_3c_2d_4 + a_2b_3c_1d_4 - a_2b_1c_3d_4 + a_3b_1c_2d_4 - a_3b_2c_1d_4 + a_1b_4c_2d_3 \\ &- a_1b_2c_4d_3 + a_4b_2c_1d_3 - a_4b_1c_2d_3 + a_2b_1c_4d_3 - a_2b_4c_1d_3 + a_3b_4c_1d_2 \\ &- a_3b_1c_4d_2 + a_4b_1c_3d_2 - a_4b_3c_1d_2 + a_1b_3c_4d_2 - a_1b_4c_3d_2 + a_2b_4c_3d_1 \\ &- a_2b_3c_4d_1 + a_4b_3c_2d_1 - a_4b_2c_3d_1 + a_3b_2c_4d_1 - a_3b_4c_2d_1 = 0. \end{aligned}$$

8. There is no difficulty in extending to any number of equations the process here employed; and the reader will observe that the general expression for a determinant is $\Sigma \pm a_1b_2c_3d_4$, &c., where each product must include all the varieties of the n letters and of the n suffixes, without repetition or omission, and the determinant contains all possible such products which can be formed. With regard to the sign to be affixed to each element of the determinant, the following is the rule :—"*The sign + or – is affixed to each product according as it is derived from the first term by an even or odd number of permutations of suffixes.*" Thus, in the last example, the second term $a_1b_3c_2d_4$ differs from the first only by a permutation of the suffixes of b and c; it therefore has an opposite sign. The third term, $a_2b_3c_1d_4$, differs from the second by a permutation of the suffixes of a and c; it therefore has an opposite sign: but it has the same sign with the first term, since it can only be derived from it by *twice* permuting suffixes.

Ex.—In the determinant $(a_1b_2c_3d_4e_5)$, what sign is to be affixed to the element $a_3b_5c_2d_1e_4$?

From the first term, permuting the suffixes of a and c, we get $a_3b_2c_1d_4e_5$, the first constituent of which is the same as that in the given term: next permuting the suffixes of b and e, we get $a_3b_5c_1d_4e_2$, which has two constituents the same as the given term: next, permuting c and e, we get $a_3b_5c_2d_4e_1$: lastly, permuting d and e, we get the given term $a_3b_5c_2d_1e_4$. Since, then, there has been an even number (four) of permutations, the sign of the term is +. In fact, the signs of the series of terms are

$$a_1b_2c_3d_4e_5 - a_3b_2c_1d_4e_5 + a_3b_5c_1d_4e_2 - a_3b_5c_2d_4e_1 + a_3b_5c_2d_1e_4.$$

9. *A cyclic interchange of suffixes alters the sign when the number of terms in the product is even; but not so when the num-*

ber of terms is odd. Thus a_2b_1, being got from a_1b_2 by one interchange of suffixes, has a different sign; but $a_2b_3c_1$ has the same sign with $a_1b_2c_3$ from which it is derived by a double permutation. For, changing the suffixes of a and b, $a_1b_2c_3$ becomes $a_2b_1c_3$, and changing the suffixes of b and c, this again becomes $a_2b_3c_1$. In like manner $a_2b_3c_4d_1$ has an opposite sign to $a_1b_2c_3d_4$, being derived from it by a triple permutation, viz., through the steps $a_2b_1c_3d_4$, $a_2b_3c_1d_4$, $a_2b_3c_4d_1$.

10. We are now in a position to replace our former definition of a determinant by another, which we make the foundation of the subsequent theory. In fact, since a determinant is only a function of its constituents a_1, b_1, c_1, &c., and does not contain the variables x, y, z, &c., it is obviously preferable to give a definition which does not introduce any mention of equations between these quantities x, y, z.

*Let there be n^2 quantities arrayed in a square of n columns and n rows, then the determinant of these quantities is the sum with proper signs (as explained, Art. 9) of all possible products of n constituents, one constituent being taken from each horizontal and each vertical row. It is very common to write the constituents of a determinant with a double suffix, the first suffix denoting the row, and the second the column, to which the constituent belongs. Thus the determinant of the third order would be written—

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3} \\ a_{2,1}, & a_{2,2}, & a_{2,3} \\ a_{3,1}, & a_{3,2}, & a_{3,3} \end{vmatrix}$$

or else

$$\Sigma \pm a_{1,1},\ a_{2,2},\ a_{3,3},$$

where in the sum the suffixes are interchanged in all possible ways.

* We might have commenced with this definition of a determinant, the preceding articles being unnecessary to the scientific development of the theory. We have thought, however, that the illustrations there given would make the general theory more intelligible; and also that the importance of the study of determinants would more clearly appear, when it had been shown that every elimination of the variables from a system of equations of the first degree, and every solution of such a system, gives rise to determinants, such systems of equations being of constant occurrence in every department of pure and applied mathematics.

Again, the preceding notation is sometimes modified by the omission of the letter a, and the determinant is written

$$\begin{vmatrix} (1,1), & (1,2), & (1,3) \\ (2,1), & (2,2), & (2,3) \\ (3,1), & (3,2), & (3,3) \end{vmatrix}$$

*Although these notations have several advantages, yet they are so cumbrous that we have preferred the method employed in the preceding Examples, of writing all the constituents in the same row with the same letter, and those in the same column with the same suffix.

Constituents such as (12), (21), are said to be *conjugate* to each other, that is, when the place which each occupies in the horizontal rows is the same as that which the other occupies in the vertical rows. A determinant is said to be *symmetrical* when the conjugate constituents are equal to each other; for example,

$$\begin{vmatrix} a, & b, & c \\ b, & d, & e \\ c, & e, & f \end{vmatrix}$$

LESSON II.

MULTIPLICATION OF DETERMINANTS.

11. We have in the last Lesson given the rule for the formation of determinants, and exemplified some of their properties in particular cases. We shall in this Lesson prove these properties in general, together with some others, especially those necessary to the establishment of the fundamental theorem, that the product of two determinants can be expressed as a determinant.

The value of a determinant is not altered if the vertical rows be written horizontally, and vice versâ (see Arts. 2, 5).

This follows immediately from the law of formation (Art. 10),

* For another notation for determinants (Mr. Sylvester's), see note on "Commutants."

which is perfectly symmetrical with respect to the columns and rows. One of the principal advantages of the notation with double suffixes is that it exhibits most distinctly the symmetry which exists between the horizontal and vertical lines.

12. *If any two rows (or two columns) be interchanged, the sign of the determinant is altered.*

For the effect of the change is evidently a single permutation of two of the letters (or of two of the suffixes), which by the law of formation causes a change of sign.

13. *If two rows (or if two columns) be identical, the determinant vanishes.*

For these two rows being interchanged, we ought (Art. 12) to have a change of sign: but the interchange of two identical lines can produce no change in the value of the determinant. Its value, then, does not alter when its sign is changed; that is to say, it is $= 0$.

This theorem also follows immediately from the definition of a determinant, as the result of elimination between n linear equations. For that elimination is performed by solving for the variables from $n-1$ of the equations, and substituting the values so found in the n^{th}. But if this n^{th} equation be the same as one of the others, it must vanish identically when these values are substituted in it.

14. *If every constituent in any row (or in any column) be multiplied by the same factor, then the determinant is multiplied by that factor.*

This follows at once from the fact that every term in the expansion of the determinant contains as a factor, one, and but one, constituent belonging to the same row or to the same column.

Thus, for example, since every element of the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

contains either a_1, a_2, or a_3, the determinant can be written in the form $a_1A_1 + a_2A_2 + a_3A_3$ (where neither A_1, A_2, nor A_3 contains any

constituent from the a column); and if a_1, a_2, a_3 be each multiplied by the same factor ρ, the determinant will be multiplied by that factor.

15. *If every constituent in any row (or in any column) be resolvable into the sum of two others, the determinant is resolvable into the sum of two others.*

This follows from the principle used in the last Article. Thus, if in the Example there given, we write $a_1 + \alpha_1$ for a_1; $b_1 + \beta_1$ for b_1; $c_1 + \gamma_1$ for c_1; then the determinant would become

$$(a_1 + \alpha_1) A_1 + (b_1 + \beta_1) B_1 + (c_1 + \gamma_1) C_1$$
$$= \{a_1 A_1 + b_1 B_1 + c_1 C_1\} + \{\alpha_1 A_1 + \beta_1 B_1 + \gamma_1 C_1\}.$$

Thus we have

$$\begin{vmatrix} a_1 + \alpha_1, & a_2, & a_3 \\ b_1 + \beta_1, & b_2, & b_3 \\ c_1 + \gamma_1, & c_2, & c_3 \end{vmatrix} = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1, & a_2, & a_3 \\ \beta_1, & b_2, & b_3 \\ \gamma_1, & c_2, & c_3 \end{vmatrix}$$

In like manner, if the terms in any one column were each the sum of any number of others, the determinant could be resolved into the same number of others.

16. If again in the preceding, the terms in the second column were also each the sum of others (if, for instance, we were to write for a_2, $a_2 + \alpha_2$; for b_2, $b_2 + \beta_2$; for c_2, $c_2 + \gamma_2$), then each of the determinants on the right-hand side of the last equation could be resolved into the sum of others; and we see, without difficulty, that

$$(a_1 + \alpha_1,\ b_2 + \beta_2, c_3) = (a_1 b_2 c_3) + (a_1 \beta_2 c_3) + (\alpha_1 b_2 c_3) + (\alpha_1 \beta_2 c_3).$$

And if each of the constituents in the first column could be resolved into the sum of m others, and each of those of the second into the sum of n others, then the determinant could be resolved into the sum of mn others. For we should first, as in the last Article, resolve the determinant into the sum of m others, by taking, instead of the first column, each one of the m partial columns; and then, in like manner, resolve each of these into n others, by dealing similarly with the second column. And so, in general, if each of the constituents of a determinant consist of the sum of a number of terms, so that each of the columns can

be resolved into the sum of a number of partial columns (the first into m partial columns, the second into n, the third into p, &c.), then the determinant is equal to the sum of all the determinants which can be formed by taking, instead of each column, one of its partial columns, and the number of such determinants will be the product of the numbers m, n, p, &c.

17. We come now to one of the most important theorems on the subject of determinants.

The product of two determinants is the determinant whose constituents are the sum of the products of the constituents in any row of one by the corresponding constituents in any row of the other.

For example, the product of the determinants $(a_1b_2c_3)$ and $(\alpha_1\beta_2\gamma_3)$ is

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1, & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2, & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1, & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2, & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1, & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2, & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

The proofs which we shall give for this particular case will apply equally in general. Since the constituents of the determinant just written are each the sum of three terms, the determinant can (by Art. 16) be resolved into the sum of the 27 determinants, obtained by taking any one partial column of the first, second, and third column. We need not write down the whole 27, but give two or three specimen terms.

$$\begin{vmatrix} a_1\alpha_1, & a_1\alpha_2, & a_1\alpha_3 \\ a_2\alpha_1, & a_2\alpha_2, & a_2\alpha_3 \\ a_3\alpha_1, & a_3\alpha_2, & a_3\alpha_3 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1, & b_1\beta_2, & c_1\gamma_3 \\ a_2\alpha_1, & b_2\beta_2, & c_2\gamma_3 \\ a_3\alpha_1, & b_3\beta_2, & c_3\gamma_3 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1, & c_1\gamma_2, & b_1\beta_3 \\ a_2\alpha_1, & c_2\gamma_2, & b_2\beta_3 \\ a_3\alpha_1, & c_3\gamma_2, & b_3\beta_3 \end{vmatrix} + \&c.$$

Now it will be observed that in all these determinants every column has a common factor, which (Art. 14) may be taken out as a multiplier of the entire determinant. The specimen terms already given may therefore be written in the form

$$\alpha_1\alpha_2\alpha_3 \begin{vmatrix} a_1, & a_1, & a_1 \\ a_2, & a_2, & a_2 \\ a_3, & a_3, & a_3 \end{vmatrix} + \alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} + \alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1, & c_1, & b_1 \\ a_2, & c_2, & b_2 \\ a_3, & c_3, & b_3 \end{vmatrix}$$

But the first of these determinants vanishes, since two columns are the same; the second is the determinant $(a_1b_2c_3)$; and the third

(Art. 12) is $= -(a_1b_2c_3)$. In like manner, every other partial determinant will vanish which has two columns the same; and it will be found that every determinant which does not vanish will be $(a_1b_2c_3)$, while the factors which multiply it will be the elements of the determinant $(\alpha_1\beta_2\gamma_3)$.

It would have been equally possible to break up the determinant into a series of terms, every one of which would have been the determinant $(\alpha_1\beta_2\gamma_3)$ multiplied by one of the elements of $(a_1b_2c_3)$.

18. On account of the importance of this theorem, we give another proof, founded on our first definition of a determinant.

The determinant which we examined in the last Article is the result of elimination between the equations

$$(a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1)x + (a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2)y + (a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3)z = 0,$$
$$(a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1)x + (a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2)y + (a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3)z = 0,$$
$$(a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1)x + (a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2)y + (a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3)z = 0.$$

Now if we write

$$\alpha_1x + \alpha_2y + \alpha_3z = X,$$
$$\beta_1x + \beta_2y + \beta_3z = Y,$$
$$\gamma_1x + \gamma_2y + \gamma_3z = Z,$$

the three preceding equations may be written

$$a_1X + b_1Y + c_1Z = 0,$$
$$a_2X + b_2Y + c_2Z = 0,$$
$$a_3X + b_3Y + c_3Z = 0,$$

from which eliminating X, Y, Z, we see at once that $(a_1b_2c_3)$ must be a factor in the result. But also a system of values of x, y, z can be found to satisfy the three given equations, provided that a system can be found to satisfy simultaneously the equations $X = 0$, $Y = 0$, $Z = 0$. Hence $(\alpha_1\beta_2\gamma_3) = 0$, which is the condition that the latter should be possible, is also a factor in the result. And since we can see without difficulty that the degree of the result in the coefficients is exactly the same as that of the product of these quantities, the result is $(a_1b_2c_3)\,(\alpha_1\beta_2\gamma_3)$.

It appears from the present Article that the theorem concerning the multiplication of determinants can be expressed in

the following form, in which we shall frequently employ it:—
If a system of equations

$$a_1X + b_1Y + c_1Z = 0, \quad a_2X + b_2Y + c_2Z = 0, \quad a_3X + b_3Y + c_3Z = 0$$

be transformed by the substitutions

$$X = \alpha_1x + \alpha_2y + \alpha_3z, \quad Y = \beta_1x + \beta_2y + \beta_3z, \quad Z = \gamma_1x + \gamma_2y + \gamma_3z,$$

then the determinant of the transformed system will be equal to $(a_1b_2c_3)$ *the determinant of the original system, multiplied by* $(\alpha_1\beta_2\gamma_3)$, which we shall call *the modulus of transformation.*

19. The theorems of the last Articles may be extended as follows:—We might have two sets of constituents, the number of rows being different from the number of columns; for example—

$$\left\|\begin{matrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{matrix}\right\| \qquad \left\|\begin{matrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{matrix}\right\|$$

and from these we could form, in the same manner as in the last Articles, the determinant

$$\left|\begin{matrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1, & a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 \\ a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2, & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 \end{matrix}\right|$$

whose value we purpose to investigate.

Now, first, let the number of columns be greater than the number of rows, as in the example just written; so that each constituent of the new determinant is the sum of a number of terms greater than the number of rows: then proceeding as in Art. 17, the value of the determinant is

$$\left|\begin{matrix} a_1\alpha_1, & a_2\alpha_1 \\ a_1\alpha_2, & a_2\alpha_2 \end{matrix}\right| + \left|\begin{matrix} a_1\alpha_1, & b_2\beta_1 \\ a_1\alpha_2, & b_2\beta_2 \end{matrix}\right| + \&c.$$

$= (a_1b_2)\,(\alpha_1\beta_2) + (a_1c_2)\,(\alpha_1\gamma_2) + (b_1c_2)\,(\beta_1\gamma_2)$. That is to say, *the new determinant is the sum of the product of every possible determinant which can be formed out of the one set of constituents by the corresponding determinant formed out of the other set of constituents.*

20. But in the second place, let the number of rows exceed the number of columns. Thus, from the two sets of constituents

$$\left\| \begin{matrix} a_1, & b_1 \\ a_2, & b_2 \\ a_3, & b_3 \end{matrix} \right\| \qquad \left\| \begin{matrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \\ \alpha_3, & \beta_3 \end{matrix} \right\|$$

let us form the determinant

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1, & a_2\alpha_1 + b_2\beta_1, & a_3\alpha_1 + b_3\beta_1 \\ a_1\alpha_2 + b_1\beta_2, & a_2\alpha_2 + b_2\beta_2, & a_3\alpha_2 + b_3\beta_2 \\ a_1\alpha_3 + b_1\beta_3, & a_2\alpha_3 + b_2\beta_3, & a_3\alpha_3 + b_3\beta_3 \end{vmatrix}$$

Then when we proceed to break this up into partial determinants in the manner already explained, it will be found impossible to form any partial determinant which shall not have two columns the same. *The determinant therefore will vanish identically.*

21. A useful particular case of Art. 17 is, that *the square of a determinant is a symmetrical determinant* (see Art. 10). Thus the square of $(a_1b_2c_3)$ is

$$\begin{vmatrix} a_1^2 + b_1^2 + c_1^2, & a_1a_2 + b_1b_2 + c_1c_2, & a_1a_3 + b_1b_3 + c_1c_3 \\ a_1a_2 + b_1b_2 + c_1c_2, & a_2^2 + b_2^2 + c_2^2, & a_2a_3 + b_2b_3 + c_2c_3 \\ a_1a_3 + b_1b_3 + c_1c_3, & a_2a_3 + b_2b_3 + c_2c_3, & a_3^2 + b_3^2 + c_3^2 \end{vmatrix}$$

Again, it appears by Art. 19 that the sum of the squares of the determinants $(a_1b_2)^2 + (b_1c_2)^2 + (c_1a_2)^2$ is the determinant

$$\begin{vmatrix} a_1^2 + b_1^2 + c_1^2, & a_1a_2 + b_1b_2 + c_1c_2 \\ a_1a_2 + b_1b_2 + c_1c_2, & a_2^2 + b_2^2 + c_2^2 \end{vmatrix}$$

Ex. 1. Given n quantities α, β, γ, &c., to find the value of

$$\begin{vmatrix} 1, & 1, & 1, & 1, & \text{\&c.} \\ \alpha, & \beta, & \gamma, & \delta, & \text{\&c.} \\ \alpha^2, & \beta^2, & \gamma^2, & \delta^2, & \text{\&c.} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha^{n-1}, & \beta^{n-1}, & \gamma^{n-1}, & \delta^{n-1}, & \text{\&c.} \end{vmatrix}$$

It is evident (Art. 13) that this determinant would vanish if $\alpha = \beta$, therefore $\alpha - \beta$ is a factor in it. In like manner so is every other difference between any two of the quantities α, β, &c. The determinant is therefore

$$= \pm (\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)(\gamma - \delta) \text{ \&c.} = 0.$$

For the determinant is either equal to this product, or to the product multiplied by some factor. But there can be no factor containing α, β, &c., since the product contains $\alpha^{n-1}, \beta^{n-1}$, &c.; and the determinant can contain no higher power of α, β, &c., and by comparing the coefficients of α^{n-1} it will be seen that the determinant contains no numerical factor.

Ex. 2. In the theory of equations it is important to express the product of the squares of the differences of the roots, and the last example enables us to do this as a determinant. For if we form the square of the determinant discussed in the last Article, we obtain

$$\begin{vmatrix} s_0, & s_1, & s_2 \dots & s_{n-1} \\ s_1, & s_2, & s_3 \dots & s_n \\ s_2, & s_3, & s_4 \dots & s_{n+1} \\ \dots & \dots & \dots & \dots \\ s_{n-1}, & s_n, & s_{n+1} \dots & s_{2n-2} \end{vmatrix}$$

where s_p denotes the sum of the p^{th} powers of the quantities α, β, &c.

Ex. 3. In like manner it is proved by Art. 19 that the determinant

$$\begin{vmatrix} s_0, & s_1 \\ s_1, & s_2 \end{vmatrix} = \Sigma(\alpha - \beta)^2,$$

$$\begin{vmatrix} s_0, & s_1, & s_2 \\ s_1, & s_2, & s_3 \\ s_2, & s_3, & s_4 \end{vmatrix} = \Sigma(\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2.$$

We thus form a series of determinants, the last of which is the product of the squares of the differences of α, β, &c.; all similar determinants beyond this vanishing identically by Art. 20. This series of determinants is of great importance in the theory of algebraic equations.

LESSON III.

MINOR AND RECIPROCAL DETERMINANTS.

22. If in any determinant we erase any number of rows and the same number of columns, the determinant formed with the remaining rows and columns is called a *minor* of the given determinant. The minors formed by erasing one row and one column may be called first minors; those formed by erasing two rows and columns, second minors; and so on.

We have in the last Lesson employed the principle, that if $a_1b_1c_1$ &c., be the constituents of any one row of a determinant, then the determinant is of the form $a_1A_1 + b_1B_1 + c_1C_1 + $ &c., where A_1, B_1, C_1, &c., do not contain a_1, b_1, c_1, &c. We wish now to go into a little detail as to the value of these quantities A_1, B_1, &c. And we say that A_1 is the minor obtained by erasing the line and column which contain a_1. For every ele-

ment of the original determinant which contains a_1 can contain no other constituent from the column a or the line (1), and a_1 must be multiplied by all possible combinations of products of $n-1$ constituents, taken one from each of the other rows and columns. But the aggregate of these form the minor determinant just defined.

In like manner the determinant may be written in the form $a_1A_1 + a_2A_2 + a_3A_3 + \&c.$, where A_2 is the minor formed by erasing the line and column which contain a_2.

It is evident that A_1 is the differential coefficient of the original determinant taken with respect to a_1.

23. These minors and the constituents are connected by a series of identical relations, viz.:

$$b_1A_1 + b_2A_2 + b_3A_3 + \&c. = 0,$$
$$c_1A_1 + c_2A_2 + c_3A_3 + \&c. = 0, \&c.$$

For since the determinant is equal to $a_1A_1 + a_2A_2 + \&c.$, and since $A_1, A_2, \&c.$, do not contain $a_1, a_2, \&c.$, therefore $b_1A_1 + b_2A_2 + \&c.$, is what the determinant would become if we were to make in it $a_1 = b_1$, $a_2 = b_2$, &c.; but the determinant would then have two columns identical, and would therefore vanish (Art. 13).

24. We can now briefly write the solution of a system of equations—

$$a_1x + b_1y + c_1z + \&c. = \xi,$$
$$a_2x + b_2y + c_2z + \&c. = \eta,$$
$$a_3x + b_3y + c_3z + \&c. = \zeta, \&c.,$$

for, multiply the first by A_1, the second by A_2, &c., and add, and the coefficients of y, z, &c., will vanish identically, while the coefficient of x will be $a_1A_1 + a_2A_2 + \&c.$, which is the determinant formed out of the coefficients on the left-hand side of the equation, which we shall call Δ. Thus we get

$$\Delta x = A_1\xi + A_2\eta + A_3\zeta + \&c.$$
$$\Delta y = B_1\xi + B_2\eta + B_3\zeta + \&c.$$
$$\Delta z = C_1\xi + C_2\eta + C_3\zeta + \&c., \&c.$$

25. The *reciprocal* of a given determinant is the determinant

whose constituents are the minors corresponding to each constituent of the given one. Thus the reciprocal of $(a_1b_2c_3)$ is

$$\begin{vmatrix} A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \\ A_3, & B_3, & C_3 \end{vmatrix}$$

where A_1, B_1, &c., have the meaning already explained. If we call this reciprocal Δ', and multiply it by the original determinant Δ, by the rule of Art. 17 we get

$$\begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1, & a_2A_1 + b_2B_1 + c_2C_1, & a_3A_1 + b_3B_1 + c_3C_1 \\ a_1A_2 + b_1B_2 + c_1C_2, & a_2A_2 + b_2B_2 + c_2C_2, & a_3A_2 + b_3B_2 + c_3C_2 \\ a_1A_3 + b_1B_3 + c_1C_3, & a_2A_3 + b_2B_3 + c_2C_3, & a_3A_2 + b_3B_3 + c_3C_3 \end{vmatrix}$$

But (Art. 23) $a_1A_1 + bB_{11} + c_1C_1 = \Delta$, $a_1A_2 + b_1B_2 + c_1C_2 = 0$, &c. This determinant, therefore, reduces to

$$\begin{vmatrix} \Delta, & 0, & 0, \\ 0, & \Delta, & 0, \\ 0, & 0, & \Delta, \end{vmatrix} = \Delta^3.$$

Hence $(a_1b_2c_3)(A_1B_2C_3) = (a_1b_2c_3)^3 \therefore (A_1B_2C_3) = (a_1b_2c_3)^2$. And in general, $\Delta'\Delta = \Delta^n \therefore \Delta' = \Delta^{n-1}$.

26. If we take the second system of equations in Art. 24, and solve these back again for ξ, η, &c., in terms of Δx, Δy, &c. we get

$$\Delta'\xi = \mathrm{a}_1\Delta x + \mathrm{b}_1\Delta y + \mathrm{c}_1\Delta z + \&c.,$$

where a_1, b_1, c_1 are the minors of the reciprocal determinant. But these values for ξ, η, ζ, &c. must be identical with the expressions originally given; hence remembering that $\Delta' = \Delta^{n-1}$, we get, by comparison of coefficients,

$$\mathrm{a}_1 = \Delta^{n-2}a_1, \quad \mathrm{b}_1 = \Delta^{n-2}b_1, \quad \mathrm{c}_1 = \Delta^{n-2}c_1, \&c.,$$

which express, in terms of the original coefficients, the first minors of the reciprocal determinant.

27. We have seen that, considering any one column a of a determinant, every element contains as a factor a constituent from that column, and therefore the determinant can be written

in the form $\Sigma a_p A_p$. In like manner, considering any two columns a, b of the determinant, it can be written in the form $\Sigma(a_p b_q) A_{p,q}$, where the sum $\Sigma(a_p b_q)$ is intended to express all possible determinants which can be formed by taking two rows of the given two columns.

For every element of the determinant contains as factors, a constituent from the column a, and another from the column b; and any term $a_p b_q c_r d_s$ &c., must, by the rule of signs, be accompanied by another, $-a_q b_p c_r d_s$ &c. Hence we see that the form of the determinant is $\Sigma(a_p b_q) A_{p,q}$; and by the same reasoning as in Art. 22 we see that the multiplier $A_{p,q}$ is the minor formed by omitting the two rows and columns in which a_p, b_q occur.

In like manner, considering any p columns of the determinant, it can be expressed as the sum of all possible determinants that can be formed by taking any p rows of the selected columns, and multiplying the minor formed with them, by the *complemental* minor; that is to say, the minor formed by erasing these rows and columns. For example,

$$\begin{aligned}(a_1b_2c_3d_4e_5)\\ &= (a_1b_2)(c_3d_4e_5) - (a_1b_3)(c_2d_4e_5) + (a_1b_4)(c_2d_3e_5) - (a_1b_5)(c_3d_4e_2)\\ &+ (a_2b_3)(c_1d_4e_5) - (a_2b_4)(c_1d_3e_5) + (a_2b_5)(c_1d_3e_4) + (a_3b_4)(c_1d_2e_5)\\ &- (a_3b_5)(c_1d_2e_4) + (a_4b_5)(c_1d_2e_3).\end{aligned}$$

The sign of each term in the above is determined without difficulty by the rule of signs (Art. 8).

28. The theorem of Art. 26 may be extended as follows:—*Any minor of the order p which can be formed out of the inverse constituents A_1, B_1, &c., is equal to the complementary of the corresponding minor of the original determinant, multiplied by the $(p-1)^{st}$ power of that determinant.*

For example, in the case where the original determinant is of the fifth order,

$$(A_1B_2) = \Delta(c_3d_4e_5), \quad (A_1B_2C_3) = \Delta^2(d_4e_5), \text{ \&c.}$$

The method in which this is proved in general will be sufficiently understood from the proof of the first Example. We have

$$\Delta x = A_1\xi + A_2\eta + A_3\zeta + A_4\omega + A_5 v,$$
$$\Delta y = B_1\xi + B_2\eta + B_3\zeta + B_4\omega + B_5 v.$$

Therefore

$$\Delta B_2 x - \Delta A_2 y = (A_1B_2)\xi + (A_3B_2)\zeta + (A_4B_2)\omega + (A_5B_2)\upsilon.$$

But we can get another expression for x in terms of the same five quantities, $y, \xi, \zeta, \omega, \upsilon$. For, consider the original equations,

$$\begin{aligned} \xi &= a_1x + b_1y + c_1z + d_1w + e_1u, \\ \zeta &= a_3x + b_3y + c_3z + d_3w + e_3u, \\ \omega &= a_4x + b_4y + c_4z + d_4w + e_4u, \\ \upsilon &= a_5x + b_5y + c_5z + d_5w + e_5u, \end{aligned}$$

and eliminate z, w, u, when we get

$$(a_1c_3d_4e_5)x + (b_1c_3d_4e_5)y = (c_3d_4e_5)\xi - (c_4d_5e_1)\zeta + (c_5d_1e_3)\omega - (c_1d_3e_4)\upsilon;$$

and since $(a_1c_3d_4e_5)$ is by definition $= B_2$, comparing these equations with those got already, we find $(A_1B_2) = \Delta(c_3d_4e_5)$, &c. Q. E. D.

LESSON IV.

DEGREE OF ELIMINANTS.

29. THE result of eliminating n variables between n homogeneous* equations of any degree is called the *eliminant* of these equations. We commence with the theory of the equations between two variables—

$$\begin{aligned} \phi &= a_0x^m + a_1x^{m-1}y + a_2x^{m-2}y^2 + \&c. = 0, \\ \psi &= b_0x^n + b_1x^{n-1}y + b_2x^{n-2}y^2 + \&c. = 0. \end{aligned}$$

The degree of the eliminant in the coefficients is most easily obtained by the method of elimination by symmetric functions, an explanation of which we give here for completeness, although it

* We use homogeneous equations for symmetry, but it is evident that by dividing each equation by the highest power of any one variable, n homogeneous equations between n variables can be reduced to n non-homogeneous equations between $n-1$ variables.

may be found in most books on Algebra. If we divide the equations just written by a_0y^m, b_0y^n respectively, and write

$$\frac{x}{y}=t,\ \frac{a_1}{a_0}=p,\ \frac{b_1}{b_0}=p',\ \frac{a_2}{a_0}=q,\ \&c.,$$

we reduce them to non-homogeneous functions of a single variable—

$$t^m+pt^{m-1}+qt^{m-2}+\&c.=0,$$
$$t^n+p't^{n-1}+q't^{n-2}+\&c.=0.$$

We discuss them first in the latter form, because it is that with which the learner is most likely to be familiar. Let the roots of the first equation solved for t be α, β, γ, &c.; and those of the second α', β', γ', &c.

30. The eliminant is the condition that the two equations should have a common root. If this be the case, when we solve the first equation for t, and substitute the roots found, α, β, γ, &c., in the second equation, some one of the results $\psi(\alpha)$, $\psi(\beta)$, &c., must vanish, and therefore the product of all must be sure to vanish. But this product is a symmetric function of the roots of the first equation, and therefore can be expressed in terms of its coefficients, in which state it is the eliminant required. The rule, then, for elimination by this method, is to take the m factors,

$$\psi(\alpha)=\alpha^n+p'\alpha^{n-1}+q'\alpha^{n-2}+\&c.$$
$$\psi(\beta)=\beta^n+p'\beta^{n-1}+q'\beta^{n-2}+\&c.$$
$$\psi(\gamma)=\gamma^n+p'\gamma^{n-1}+q'\gamma^{n-2}+\&c.,\ \&c.,$$

to multiply all together, and then substitute for the symmetric functions $(\alpha\beta\gamma)^n$, &c., their values in terms of the coefficients of the first equation.

Ex. To eliminate t between $t^2+pt+q=0$, $t^2+p't+q'=0$.
Multiplying $(\alpha^2+p'\alpha+q')(\beta^2+p'\beta+q')$ we get

$$\alpha^2\beta^2+p'\alpha\beta(\alpha+\beta)+p'^2\alpha\beta+q'(\alpha^2+\beta^2)+p'q'(\alpha+\beta)+q'^2,$$

and then substituting $\alpha+\beta=-p$, $\alpha\beta=q$, we get

$$q^2-pp'q+p'^2q+q'(p^2-2q)-p'q'p+q'^2=0,$$

or

$$(q-q')^2+(p-p')(pq'-p'q)=0.$$

31. We obtain in this way the same result (or at least results differing only in sign), whether we substitute the roots of the first equation in the second, or those of the second in the first.

In other words, the eliminant may be written at pleasure in either of the forms—

$$\phi(\alpha')\,\phi(\beta')\,\phi(\gamma')\ \&c.,\ \text{ or }\ \psi(\alpha)\,\psi(\beta)\,\psi(\gamma)\ \&c.$$

For, remembering that

$$\phi(t) = (t-\alpha)(t-\beta)(t-\gamma)\ \&c.,$$

the first form is

$$(\alpha'-\alpha)(\alpha'-\beta)(\alpha'-\gamma)\ \&c.\ (\beta'-\alpha)(\beta'-\beta)(\beta'-\gamma)\ \&c.;$$

and the second is

$$(\alpha-\alpha')(\alpha-\beta')(\alpha-\gamma')\ \&c.\ (\beta-\alpha')(\beta-\beta')(\beta-\gamma')\ \&c.$$

In either case we get the product of all possible differences between a root of the first and one of the second equation; and the two products can at most differ in sign.

32. If the equations had been given in the form originally written, $a_0x^m + a_1x^{m-1}y + \&c.$, $b_0x^n + b_1x^{n-1}y + \&c.$, the eliminant could be obtained from the result of Art. 30, by writing for p, q, p', q', &c., their values, $\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{b_1}{b_0}, \frac{b_2}{b_0}$, &c. We should then, for symmetry, clear of fractions, by multiplying by the highest power of a_0 or b_0 in any denominator. Thus the eliminant of $a_0x^2 + a_1xy + a_2y^2$, $b_0x^2 + b_1xy + b_2y^2$, obtained in this manner from the result in the Example, Art. 30, is

$$(a_0b_2 - a_2b_0)^2 + (a_1b_0 - a_0b_1)(a_1b_2 - a_2b_1) = 0.$$

The eliminant is always a homogeneous function of the coefficients of either equation. For before we cleared of fractions it was evidently a homogeneous function of the degree 0, and it of course remains homogeneous when every term is multiplied by the same quantity.*

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* As in what follows we almost always use homogeneous equations, both for the sake of symmetry, and in order to preserve analogy with equations between any number of variables, it may be well to show how we might avail ourselves of the known properties of algebraic equations, without the necessity of transforming in each case to the form $t^m + pt^{m-1} + \&c.$

If the equation in the latter form is satisfied by any root $t = \alpha$, then the equation in the form $a_0x^m + \&c.$ is satisfied by any system of values x', y', provided only that we have $x' = \alpha y'$, since it is manifest that we are only concerned with the ratio $x' : y'$. Again, since we know that the equation in the form $t^m + \&c.$ is resolvable into the

33. *The eliminant of two equations of the m^{th} and n^{th} degrees respectively is of the n^{th} degree in the coefficients of the first, and of the m^{th} in the coefficients of the second.*

For it may be written either as the product of m factors, $\psi(\alpha)\,\psi(\beta)\,\psi(\gamma)$ &c., each containing the coefficients of the second equation in the first degree, or else, as the product of n factors, $\phi(\alpha')\,\phi(\beta')\,\phi(\gamma')$ &c., each containing in the first degree the coefficients of the first equation.

34. If each of the roots α, β, α', β', &c., be multiplied by the same factor ρ, then, since each of the mn differences $\alpha-\alpha'$ (see Art. 31) is multiplied by this factor ρ, the eliminant will be multiplied by ρ^{mn}. But the roots of any equation

$$a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \&\text{c.} = 0$$

are all multiplied by ρ if we multiply a_1 by ρ, a_2 by ρ^2, a_3 by ρ^3, &c. It follows, then, that if in the eliminant we substitute $a_1\rho$, $b_1\rho$ for a_1, b_1; $a_2\rho^2$, $b_2\rho^2$ for a_2, b_2, &c., then the effect will be that the eliminant will be multiplied by ρ^{mn}; or, in other words,

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product of factors $(t-\alpha)\,(t-\beta)\,(t-\gamma)$ &c., or $\left(\frac{x}{y}-\frac{x'}{y'}\right)\left(\frac{x}{y}-\frac{x''}{y''}\right)\left(\frac{x}{y}-\frac{x'''}{y'''}\right)$ &c., so we see that the equation a_0x^m+ &c. (which is the same equation cleared of fractions) is resolvable into the product of the factors $(y'x-x'y)\,(y''x-x''y)\,(y'''x-x'''y)$ &c. Again, as by comparing the actual product $(t-\alpha)\,(t-\beta)$ &c., with $t^m+pt^{m-1}+$ &c., we obtain expressions for the coefficients p, q, &c., in terms of the roots α, β, &c., so, in like manner, by actually multiplying $(y'x-x'y)\,(y''x-x''y)$ &c., and comparing the product with a_0x^m+&c., we get expressions for a_0, a_1 &c., in terms of x', y', &c. Thus we get $a_0=x'x''x'''$ &c., $a_1=-\Sigma y'x''x'''x''''$ &c., where in each term one y is multiplied by all the rest of the x's; $a_2=\Sigma y'y''x'''x''''$ &c.: $a_m=\pm y'y''y'''$ &c., $a_{m-1}=\mp\Sigma x'y''y'''y''''$ &c. In like manner, any expression for a symmetric function of the roots of t^m+ &c. in terms of its coefficients can be translated into an expression for a symmetric function of x', y', &c., in terms of a_0, a_1, &c., by substituting $\frac{x'}{y'}$ for α, &c., and clearing of fractions. We might then have applied directly the method of elimination explained (Art. 30) to equations in the form a_0x^m+ &c., b_0x^n+ &c. For having found the m systems of values x', y'; x'', y''; &c., which satisfy the first equation, and having substituted them in the second equation, we should multiply together the m results $(b_0x'^n+$ &c.$)\,(b_0x''^n+$ &c.$)$, &c.; and then, substituting a_0^n for $(x'x''x'''$ &c.$)^n$, &c., reduce the product to a function of coefficients only. We have preferred the process in the text, although less elegant and symmetrical, because the learner is supposed to be already familiar with the formation of symmetric functions of the roots of ordinary algebraic equations, and it would seem to throw an unnecessary difficulty in his way, if we were to use the equations in a form in which his previous knowledge would be less available.

every term of the eliminant will be multiplied by ρ^{mn}. But now it is evident that the effect of this substitution is to multiply any term in the eliminant by a power of ρ equal to the sum of all the suffixes in that term. Thus, for example, take the term $a_1a_2b_0b_1$ in the Example, Art. 32, and if we substitute $a_1\rho$, $a_2\rho^2$, $b_1\rho$, for a_1, a_2, b_1, the term will evidently be multiplied by ρ^4. We conclude, then, that *in the eliminant of*

$$a_0x^m + a_1x^{m-1}y + \&c. \quad b_0x^n + b_1x^{n-1}y + \&c.,$$

the sum of the suffixes in every term is constant, and $= mn$.

This result may be otherwise stated thus:—*If* a_1, b_1 *contain any new variable* z *in the first degree; if* a_2, b_2 *contain it in the second and lower degrees;* a_3, b_3 *in the third, &c.; then the eliminant will contain this variable in the* mn^{th} *degree.**

It is obvious, from symmetry, that these results would have been equally true if the equations had been written in the form $a_mx^m + a_{m-1}x^{m-1}y + \&c.$, the suffix of any coefficient corresponding to the power of x, which it multiplies, instead of to the power of y.

35. *Given two homogeneous equations between three variables, of the* m^{th} *and* n^{th} *degrees respectively, the number of systems of values of the variables which can be found to satisfy simultaneously the two equations is* mn.†

Let the two equations, arranged according to powers of x, be

$$ax^m + (by + cz)x^{m-1} + (dy^2 + eyz + fz^2)x^{m-2} + \&c. = 0,$$
$$a'x^n + (b'y + c'z)x^{n-1} + (d'y^2 + e'yz + f'z^2)x^{n-2} + \&c. = 0.$$

If now we eliminate x between these equations, since the coefficient of x^{m-1} is a homogeneous function of y and z of the first degree, that of x^{m-2} is a similar function of the second degree, and so on,—it follows from the last Article that the eliminant will be a homogeneous function of y and z of the mn^{th} degree. It

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* Or again, otherwise thus: if in the eliminant we substitute for each coefficient the term $x^\alpha y^\beta$, which it multiplies in the original equation, every term of the eliminant will be divisible by $x^{mn}y^{mn}$.

† These equations may be considered as representing two curves of the m^{th} and n^{th} degrees respectively; the geometrical interpretation of the proposition of this Article being, that two such curves intersect in mn points. The equations are reduced to ordinary Cartesian equations by making $z = 1$.

follows then that mn values of y and z* can be found which will make the eliminant $= 0$. If we substitute any one of these in the given equations, they will now have a common root when solved for x (since their eliminant vanishes); and this value of x, combined with the values of y and z already found, gives one system of values satisfying the given equations. So we plainly have in all mn such systems of values. We shall, in the next Lesson, give a method by which, when two equations have a common root, that common root can immediately be found.

Ex. To find the co-ordinates of the four points of intersection of the two conics,

$$ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy = 0, \quad a'x^2 + b'y^2 + c'z^2 + 2d'yz + 2e'zx + 2f'xy = 0.$$

Arrange the equations according to the powers of x, and eliminate that variable; then by Art. 32 the result is

$$\{(ab')y^2 + 2(ad')yz + (ac')z^2\}^2$$
$$+ 4[(af')y + (ae')z]\,[(bf')y^3 + \{(be') + 2(df')\}y^2z + \{(cf') + 2(de')\}z^2y + (ce')z^3] = 0,$$

where, as in Lesson I., we have written (ab') for $ab' - a'b$. This equation, solved for $y : z$, determines the values corresponding to the four points of intersection. Having found these, by substituting any one of them in both equations, and finding their common root, we obtain the corresponding value of $x : z$. We might have at once got the four values of $x : z$ by eliminating y between the equations, but substitution in the equations is necessary in order to find which value of y corresponds to each value of x. By making $z = 1$, what has been said is translated into the language of ordinary Cartesian co-ordinates.

36. *Any symmetric functions of the mn values which simultaneously satisfy the two equations can be expressed in terms of the coefficients of those equations.*

In order to be more easily understood, we first consider non-homogeneous equations in two variables. Then it is plain enough that we can so express symmetric functions involving either variable alone. For eliminating y, we have an equation in x, in terms of whose coefficients can be expressed all symmetric functions of the mn values of x which satisfy both equations. Similarly for y. Thus, for example, in the case of two conics; $x_{,} y_{,}$ &c., being the co-ordinates of their points of intersection, we see at once how to express such symmetric functions as

$$x_{,} + x_{,,} + x_{,,,} + x_{,,,,}, \quad y^2_{,} + y^2_{,,} + y^2_{,,,} + y^2_{,,,,}, \text{ \&c.,}$$

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* The reader will remember that when we use homogeneous equations, the *ratio* of the variables is all with which we are concerned. Thus here, z' may be taken arbitrarily, the corresponding value of y being determined by the equation in $y : z$.

and the only thing requiring explanation is how to express symmetric functions into which both variables enter, such as

$$x_{,}y_{,} + x_{,,}y_{,,} + x_{,,,}y_{,,,} + x_{,,,,}y_{,,,,}.$$

To do this, we introduce a new variable, $t = \lambda x + \mu y$, and by the help of this assumed equation eliminate both x and y from the given equations. Thus y is immediately eliminated by substituting in both its value derived from $t = \lambda x + \mu y$, and then we have two equations of the m^{th} and n^{th} degrees in x, the eliminant of which will be of the mn^{th} degree in t, and its roots will be obviously $\lambda x_{,} + \mu y_{,}$, $\lambda x_{,,} + \mu y_{,,}$, &c., where $x_{,}y_{,}$, $x_{,,}y_{,,}$ are the values of x and y common to the two equations. The coefficients of this equation in t will of course involve λ and μ. We next form the sum of the k^{th} powers of the roots of this equation in t, which must plainly be $= (\lambda x_{,} + \mu y_{,})^k + (\lambda x_{,,} + \mu y_{,,})^k + \&c.$ The coefficient, then, of λ^k in this sum will be $\Sigma x_{,}^k$: the coefficient of $\lambda^{k-1}\mu$ gives us $\Sigma x_{,}^{k-1}y_{,}$, and so on.

Little need be said in order to translate the above into the language of homogeneous equations. We see at once how to form symmetric functions involving two variables only, such as $\Sigma y_{,}z_{,,}z_{,,,}z_{,,,,}$, for these are found, as explained, Note, p. 21, from the homogeneous equation obtained on eliminating the remaining variable; the only thing requiring explanation is how to form symmetric functions involving all three variables, and this is done precisely as above, by substituting $t = \lambda x + \mu y$.

Ex. To form the symmetric functions of the co-ordinates of the four points common to two conics. The equation in the last Example gives at once

$$y_{,}y_{,,}y_{,,,}y_{,,,,} = (ac')^2 + 4(ae')(ce'); \quad z_{,}z_{,,}z_{,,,}z_{,,,,} = (ab')^2 + 4(af')(bf');$$

and from symmetry, $$x_{,}x_{,,}x_{,,,}x_{,,,,} = (bc')^2 + 4(bd')(cd'),$$

$$-\Sigma(y_{,}y_{,,}y_{,,,}z_{,,,,}) = 4\{(ac')(ad') + (af')(ce') + (ae')(cf') + 2(ae')(de')\} \text{ \&c.}$$

To take an Example of a function involving three variables, let us form

$$\Sigma(x_{,}y_{,}z^2_{,,}z^2_{,,,}z^2_{,,,,}).$$

By the preceding theory we are to eliminate between the given equations, and $t = \lambda x + \mu y$; and the required function will be half the coefficient of $\lambda\mu$ in $\Sigma(t^2_{,}z^2_{,,}z^2_{,,,}z^2_{,,,,})$. If the result of elimination be

$$At^4 + (B\lambda + C\mu)t^3z + (D\lambda^2 + E\lambda\mu + F\mu^2)t^2z^2 + \&c.$$

$$\Sigma(t^2_{,}z^2_{,,}z^2_{,,,}z^2_{,,,,}) = (B\lambda + C\mu)^2 - 2A(D\lambda^2 + E\lambda\mu + F\mu^2),$$

and $$\Sigma(x_{,}y_{,}z^2_{,,}z^2_{,,,}z^2_{,,,,}) = BC - AE.$$

By actual elimination

$$A = (ab')^2 + 4(af')(bf'), \quad B = 4\{(ba')(be') + (bd')(af') + (bf')(ad') + 2(bf')(ef')\},$$

$$C = 4\{(ab')(ad') + (ae')(bf') + (af')(be') + 2(af')(df')\},$$

$$E = 4\{(ac')(bf') + (bc')(af') - 2(ad')(fd') - 2(be')(fe') + 4(fd')(fe')\}.$$

37. *To form the eliminant of three homogeneous equations in three variables, of the m^{th}, n^{th}, and p^{th} degrees respectively.*

The vanishing of the eliminant is the condition that a system of values of x, y, z can be found to satisfy all three equations.* When this, then, is the case, if we solve for any two of the equations, and substitute successively in the remaining one the values so found for x, y, z, some one of these sets of values must satisfy that equation, and therefore the product of all the results of substitution must vanish. Let, then, x', y', z'; x'', y'', z'', &c., be the system of values which satisfy the last two equations, and which (Art. 35) are np in number, we substitute these values in the first, and multiply together the np results $\phi(x', y', z')$, $\phi(x'', y'', z'')$, &c. The product will plainly involve only symmetric functions of x', y', z', &c., which (Art. 36) can all be expressed in terms of the coefficients of the last two equations; and, when they are so expressed, it is the eliminant required.

38. *The eliminant is a homogeneous function of the np^{th} degree in the coefficients of the first equation; of the mp^{th} in those of the second; and of the mn^{th} in those of the third.*

For each of the np factors $\phi(x', y', z')$ is a homogeneous function of the first degree in the coefficients of the first equation; and the expression of the symmetric functions in terms of the coefficients only involves coefficients of the last two equations, from solving which x', y', z', &c. were obtained. The eliminant is therefore of the np^{th} degree in the coefficients of the first equation; and in like manner its degree in the coefficients of the others may be inferred.

39. *If all the coefficients in the equations which multiply the first power of one of the variables, z, be affected with a suffix* 1, *those which multiply z^2 with a suffix* 2, *and so on; the sum of all*

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* If the three equations represent curves, the vanishing of the eliminant is the condition that all three curves should pass through a common point.

the suffixes in each term of the eliminant will be equal to mnp. In other words: *If all the coefficients which multiply z contain a new variable in the first degree;—if those which multiply z^2 contain it in the second and lower degrees, and so on; then the eliminant will contain this variable in the degree mnp.*

This is proved as in Art. 34. In the first place, it is evident that if a homogeneous equation of the n^{th} degree be satisfied by values x', y', z'; and if the equation be altered by multiplying each coefficient by a power of ρ, equal to the power of z, which the coefficient multiplies, then the equation so transformed will be satisfied by the values $\rho x'$, $\rho y'$, z'; or, in general, that the result of substituting $\rho x'$, $\rho y'$, z' in the transformed equation is ρ^n times the result of substituting x', y', z' in the untransformed. Thus, take the equation $x^3 + y^3 - z^3 - z^2x - zy^2$, the transformed is $x^3 + y^3 - \rho^3z^3 - \rho^2z^2x - \rho zy^2$; and obviously the result of substituting $\rho x'$, $\rho y'$, z' in the second is ρ^3 times the result of substituting x', y', z' in the given equation. If, then, the three given equations be all transformed by multiplying each coefficient by a power of ρ equal to the power of z, which the coefficient multiplies, then it follows, that if x', y', z' be one of the system of values which satisfy the two last of the original equations, then the transformed equations will be satisfied by $(\rho x', \rho y', z')$, and the result of substituting these values in the first will be $\rho^m\phi(x', y', z')$. The eliminant, then, which is the product of np factors of the form $\phi(x', y', z')$ will be multiplied by ρ^{mnp}. If then any term in the eliminant be $a_kb_lc_m$ &c., where the suffix corresponds to the power of z, which the coefficient multiplies, since the alteration of a_k into ρ^ka_k, b_l into ρ^lb_l &c., multiplies the term by ρ^{mnp}, we must have $k + l +$ &c. $= mnp$. Q. E. D.

40. It is proved in like manner that three equations are in general satisfied by mnp common values; that any symmetric function of these values can be expressed in terms of their coefficients; and that we can form the eliminant of four equations by solving from any three of them, substituting successively in the fourth each of the systems of values so found, forming the product of the results of substitution, and then, by the method of symmetric functions, expressing the product in terms of the

coefficients of the equations. And so, in like manner, we can form the eliminant of any number of equations *which will be a homogeneous function of the coefficients of each equation of a degree equal to the product of the degrees of all the remaining equations; and if each coefficient in all the equations be affected with a suffix equal to the power of any one variable x which it multiplies, then the sum of the suffixes in every term of the eliminant will be equal to the product of the degrees of all the equations.*

LESSON V.

DETERMINATION OF COMMON ROOTS.

41. When the eliminant of any number of equations vanishes, those equations can be satisfied by a common system of values, and we purpose in this Lesson to show how that system of values can be found without actually solving the equations. Let the equations be

$$\phi = 0,\ \psi = 0,\ \chi = 0,\ \text{\&c. where } \phi = ax^m + bx^{m-1}y + cx^{m-1}z + \text{\&c.}$$

The equations are supposed to be all satisfied by the same system of values x', y', z', and their eliminant R is accordingly supposed to vanish. Now we may alter the coefficients in ϕ (a into $a + \delta a$, b into $b + \delta b$ &c.); and the transformed equation

$$ax^m + bx^{m-1}y + cx^{m-1}z + \text{\&c.} + (\delta a)x^m + (\delta b)x^{m-1}y + (\delta c)x^{m-1}z + \text{\&c.} = 0$$

will obviously still be satisfied by the values x', y', z', provided only that the variations δa, δb, &c. are connected by the single relation

$$(\delta a)x'^m + (\delta b)x'^{m-1}y' + (\delta c)x'^{m-1}z' + \text{\&c.} = 0,$$

since the first part, by hypothesis, is satisfied by x', y', z'. This transformed equation then has a system of values common with the equations ψ, χ, &c., and therefore the eliminant between it and them also vanishes. But this eliminant is obtained from the

eliminant R by altering in it a, b, &c., into $a+\delta a$, $b+\delta b$, &c., when the eliminant so transformed will be

$$R + \left\{\frac{dR}{da}\delta a + \frac{dR}{db}\delta b + \frac{dR}{dc}\delta c + \&c.\right\} + \&c. = 0.$$

$R=0$ by hypothesis, and since the variations of the coefficients may be supposed as small as we please, the terms containing the first powers of the variations must vanish separately. We have then

$$\frac{dR}{da}\delta a + \frac{dR}{db}\delta b + \&c. = 0.$$

This relation must be identical with that which we know already to be the only one which it is necessary the variations should satisfy; and therefore the several coefficients $\frac{dR}{da}$, $\frac{dR}{db}$, &c., must be respectively proportional to x'^m, $x'^{m-1}y'$, &c.: we have then $x' : y' :: \frac{dR}{da} : \frac{dR}{db}$. An equivalent expression could be obtained by taking, instead of a and b, the coefficients of any two terms which are in the ratio $x : y$; or again, by parity of reasoning, $x' : y' :: \frac{dR}{da'} : \frac{dR}{db'}$, where a', b' are the coefficients of x^n, $x^{n-1}y$ in one of the other equations.

42. We shall confirm this result by an actual examination of the values $\frac{dR}{da}$ &c. Let the results of substituting the common roots of ψ, χ, &c., in the first equation ϕ be ϕ', ϕ'', &c., then it will be remembered that $R = \phi'\phi''\phi'''$ &c., or

$$R = (ax'^m + bx'^{m-1}y' + \&c.)\ (ax''^m + bx''^{m-1}y'' + \&c.)\ \&c.,$$

therefore

$$\frac{dR}{da} = x'^m(\phi''\phi'''\ \&c.) + x''^m(\phi'\phi'''\ \&c.) + x'''^m(\phi'\phi''\ \&c.) + \&c.,$$

and if $x'y'z'$ satisfies ϕ, we have $\phi'=0$, and $\frac{dR}{da}$ reduces to $x'^m\phi''\phi'''$ &c. In like manner $\frac{dR}{db} = x'^{m-1}y'\phi''\phi'''$ &c., and so on, the several results being proportional to the terms which a, b, &c. multiply.

43. In like manner, if a, b, &c., instead of being the coefficients themselves, were any quantities of which the coefficients of ϕ were functions, but which do not enter into the equations ψ, χ, &c., it is proved by the method of either of the last Articles that $\frac{d\phi}{da} : \frac{d\phi}{db} :: \frac{dR}{da} : \frac{dR}{db}$. For, as in the last Article, we have immediately $\frac{dR}{da} = \frac{d\phi'}{da}\phi''\phi'''$ &c., and similarly for the rest. Or again, by the method of Art. 41, if a, b, c, &c., be varied so that the same system of roots continues to satisfy ϕ, we have

$$\frac{d\phi}{da}\,\delta a + \frac{d\phi}{db}\,\delta b + \frac{d\phi}{dc}\,\delta c + \text{\&c.} = 0,$$

while, because in this case the eliminant of the transformed ϕ and of the other equations continues to vanish, we have

$$\frac{dR}{da}\,\delta a + \frac{dR}{db}\,\delta b + \frac{dR}{dc}\,\delta c + \text{\&c.} = 0,$$

and these two equations must be identical.

44. If the given equations are satisfied by *two* common systems of values, that is to say, supposing we have not only $\phi' = 0$, but also $\phi'' = 0$; then it appears at once, from Art. 42, that every one of the differentials $\frac{dR}{da}$ &c. vanishes. Reciprocally, having found the condition $R = 0$, that a system of equations should have one common system of values, if we desire to find the conditions that they shall have two, we have only to equate to 0 the differentials of R, with regard to every one of the coefficients.*

To find the actual values of the common roots, we have

$$\frac{d^2R}{da^2} = x'^m x''^m (\phi'''\phi'''' \text{ \&c.}) + x'^m x'''^m (\phi''\phi'''' \text{ \&c.}) + x''^m x'''^m (\phi'^m \phi'''' \text{ \&c.})$$

which the supposition $\phi' = 0$, $\phi'' = 0$, reduces to $x'^m x''^m \phi'''\phi''''$ &c.

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* It constantly occurs in practice that a system of two conditions cannot be more simply expressed than by a system of several equations equivalent to two independent equations.

In like manner,

$$\frac{d^2R}{dadb} = \{x'^m x''^{m-1} y'' + x''^m x'^{m-1} y'\} \phi''' \phi'''' \text{ \&c.}, \quad \frac{d^2R}{db^2} = x'^{m-1} y' x''^{m-1} y'' \phi''' \phi''''.$$

If, then, we solve the quadratic equation in $\lambda : \mu$,

$$\lambda^2 \frac{d^2R}{da^2} - \lambda\mu \frac{d^2R}{dadb} + \mu^2 \frac{d^2R}{db^2} = 0,$$

the two roots will give the ratio of the terms which a and b multiply, viz., $x'^m : x'^{m-1}y'$ and $x''^m : x''^{m-1}y''$.

If the equations had three common systems of values, all the second differentials of R vanish, and the common roots are found by proceeding to the third differential coefficients, and solving a cubic equation.

45. The method of Art. 41 may be applied to the differentials of the eliminant with regard to quantities a, b, c, &c., which enter into all the equations. As before, we give these quantities variations consistent with the supposition that the eliminant still vanishes, and therefore such that

$$\frac{dR}{da}\delta a + \frac{dR}{db}\delta b + \frac{dR}{dc}\delta c + \text{\&c.} = 0.$$

Now, in the former case, where a, b, c, &c., only entered into one of the equations, a change in these quantities produced no change in the value of the common roots, since the coefficients remained constant in the other equations, whose system of common roots was therefore fixed and determinate. But this will now be no longer the case, and the common roots of the transformed equations may be different from the common roots of the original system. Let the new system of common roots be $x + \delta x$, $y + \delta y$, $z + \delta z$, &c., then the variations are connected by the relations—

$$\frac{d\phi}{da}\delta a + \frac{d\phi}{db}\delta b + \text{\&c.} + \frac{d\phi}{dx}\delta x + \frac{d\phi}{dy}\delta y + \frac{d\phi}{dz}\delta z + \text{\&c.} = 0,$$

$$\frac{d\psi}{da}\delta a + \frac{d\psi}{db}\delta b + \text{\&c.} + \frac{d\psi}{dx}\delta x + \frac{d\psi}{dy}\delta y + \frac{d\psi}{dz}\delta z + \text{\&c.} = 0,$$

$$\frac{d\chi}{da}\delta a + \frac{d\chi}{db}\delta b + \text{\&c.} + \frac{d\chi}{dx}\delta x + \frac{d\chi}{dy}\delta y + \frac{d\chi}{dz}\delta z + \text{\&c.} = 0.$$

Now, since we are using homogeneous equations, in which the *ratio* of the roots is all that we are concerned with, we may suppose one of the variables z to be the same for all the equations, and therefore may take $\delta z = 0$. From the remaining n equations we can eliminate the $n-1$ variations, δx, δy, &c., and so arrive at a relation between the variations δa, δb, &c., only, the coefficients of which must be severally proportional to $\frac{dR}{da}, \frac{dR}{db}$, &c. Thus, in the case of three variables, we have

$$\begin{vmatrix} \frac{d\phi}{da}, & \frac{d\phi}{dx}, & \frac{d\phi}{dy} \\ \frac{d\psi}{da}, & \frac{d\psi}{dx}, & \frac{d\psi}{dy} \\ \frac{d\chi}{da}, & \frac{d\chi}{dx}, & \frac{d\chi}{dy} \end{vmatrix} . \delta a + \begin{vmatrix} \frac{d\phi}{db}, & \frac{d\phi}{dx}, & \frac{d\phi}{dy} \\ \frac{d\psi}{db}, & \frac{d\psi}{dx}, & \frac{d\psi}{dy} \\ \frac{d\chi}{db}, & \frac{d\chi}{dx}, & \frac{d\chi}{dy} \end{vmatrix} . \delta b + \&c. = 0,$$

the coefficients of δa, δb, &c., in which, must be proportional to $\frac{dR}{da}, \frac{dR}{db}$, &c.

LESSON VI.

EXPRESSION OF ELIMINANTS AS DETERMINANTS.

46. The method of elimination by symmetric functions is, in a theoretical point of view, perhaps preferable to any other, it being universally applicable to equations in any number of variables: yet as it is not very expeditious in practice, and does not yield its results in the most convenient form, we shall in this Lesson give an account of some other methods; and, in particular, show that the eliminant of two equations can always be expressed as a determinant.

The eliminant of two equations of the m^{th} and n^{th} degrees respectively is obtained by Euler as follows:—If the two equations have a common factor, then we must obtain identical results

whether we multiply the first equation by the remaining $n-1$ factors of the second, or the second by the remaining $m-1$ factors of the first. If then we multiply the first by an arbitrary function of the $n-1^{st}$ degree, which, of course, introduces n arbitrary constants; if we multiply the second by an arbitrary function of the $m-1^{st}$ degree, introducing thus m constants; and if we then equate, term by term, the two equations of the $(m+n-1)^{st}$ degree so formed, we shall have $m+n$ equations, from which we can eliminate the $m+n$ introduced constants, which all enter into those equations only in the first degree; and we shall thus obtain, in the form of a determinant, the eliminant of the two given equations.

Ex. To eliminate between $ax^2+bxy+cy^2=0$, $a'x^2+b'xy+c'y^2=0$.
We are to equate, term by term,

$$(Ax+By)\,(ax^2+bxy+cy^2) \text{ and } (A'x+B'y)\,(a'x^2+b'xy+c'y^2).$$

The four resulting equations are

$$\begin{aligned} Aa \ldots\ldots - A'a' \ldots\ldots &= 0,\\ Ab+Ba-A'b'-B'a' &= 0,\\ Ac+Bb-A'c'-B'b' &= 0,\\ Bc \ldots\ldots - B'c' &= 0, \end{aligned}$$

from which eliminating A, B, A', B', the result is the determinant

$$\begin{vmatrix} a, & 0, & a', & 0 \\ b, & a, & b', & a' \\ c, & b, & c', & b' \\ 0, & c, & 0, & c' \end{vmatrix}$$

47. This method may be extended to find the conditions that the equations should have two common factors. In this case it is evident, in like manner, that we shall obtain the same result whether we multiply the first by the remaining $n-2$ factors of the second, or the second by the remaining $m-2$ factors of the first. As before, then, we multiply the first by an arbitrary function of the $n-2$ degree (introducing $n-1$ constants), and the second by an arbitrary function of the $m-2$ degree; and equating, term by term, the two equations of the $m+n-2$ degree so found, we have $m+n-1$ equations, from any $m+n-2$ of which, eliminating the $m+n-2$ introduced constants, we obtain $m+n-1$ conditions, equivalent, of course, to only two independent conditions. These conditions are simpler than the

conditions $\frac{dR}{da} = 0$ &c. found, Art. 44: for these last contain the coefficients of the equations in a degree only one less than the eliminant; while it will be found that the conditions obtained by the method of this Article are always of a degree two less than that of its eliminant.

Ex. To find the conditions that

$$ax^3 + bx^2y + cxy^2 + dy^3 = 0, \quad a'x^3 + b'x^2y + c'xy^2 + d'y^3 = 0,$$

should have two common factors. Equating

$$(Ax + By)(ax^3 + bx^2y + cxy^2 + dy^3) = (A'x + B'y)(a'x^3 + b'x^2y + c'xy^2 + d'y^3),$$

we have

$$\begin{aligned} Aa \ldots\ldots - A'a' \ldots\ldots &= 0, \\ Ab + Ba - A'b' - B'a' &= 0, \\ Ac + Bb - A'c' - B'b' &= 0, \\ Ad + Bc - A'd' - B'c' &= 0, \\ Bd \ldots\ldots - B'd' &= 0, \end{aligned}$$

from which, eliminating A, B, A', B', we have the system of determinants (for the notation used, see Art. 3),

$$\left\| \begin{matrix} a, & b, & c, & d, & 0 \\ 0, & a, & b, & c, & d \\ a', & b', & c', & d', & 0 \\ 0, & a', & b', & c', & d' \end{matrix} \right\| = 0.$$

48. Mr. Sylvester has given a method of elimination, which he calls the *dialytic* method, identical in its results with Euler's, but simpler in its application, and more easily capable of being extended. Multiply the equation of the m^{th} degree by x^{n-1}, $x^{n-2}y$, $x^{n-3}y^2$, &c.; and the second equation by x^{m-1}, $x^{m-2}y$, $x^{m-3}y^2$, &c., and we thus get $m+n$ equations, from which we can eliminate linearly the $m+n$ quantities x^{m+n-1}, $x^{m+n-2}y$, $x^{m+n-3}y^2$, &c., considered as independent unknowns. Thus in the case of two quadratics, multiply both by x and by y, and we get the equations—

$$\begin{aligned} ax^3 + bx^2y + cxy^2 \qquad\qquad &= 0, \\ ax^2y + bxy^2 + cy^3 &= 0, \\ a'x^3 + b'x^2y + c'xy^2 \qquad\qquad &= 0, \\ a'x^2y + b'xy^2 + c'y^3 &= 0, \end{aligned}$$

from which, eliminating x^3, x^2y, xy^2, y^3, we get the same determinant as before—

$$\begin{vmatrix} a, & b, & c, & \\ & a, & b, & c \\ a', & b', & c', & \\ & a', & b', & c' \end{vmatrix}$$

49. The determinants obtained by this method are inferior in simplicity to the forms obtained by the method of Art. 30. The following method, however, due to Bezout, expresses the eliminant also in the form of a determinant, and one which can be more rapidly calculated. The general method will, perhaps, be better understood if we apply it first to the particular case of the two equations of the fourth degree—

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0,\ \ a'x^4 + b'x^3y + c'x^2y^2 + d'xy^3 + e'y^4 = 0.$$

Multiplying the first by a', the second by a, and subtracting, the first term in each is eliminated, and the result, being divisible by y, gives

$$(ab')x^3 + (ac')x^2y + (ad')xy^2 + (ae')y^3 = 0.$$

Again, multiply the first by $a'x + b'y$, and the second by $ax + by$, and the two first terms in each are eliminated, and the result, being divided by y^2, gives

$$(ac')x^3 + \{(ad') + (bc')\}x^2y + \{(ae') + (bd')\}xy^2 + (be')y^3 = 0.$$

Next, multiply the first by $a'x^2 + b'xy + c'y^2$; and the second by $ax^2 + bxy + cy^2$; subtract, and divide by y^3; when we get

$$(ad')x^3 + \{(ae') + (bd')\}x^2y + \{(be') + (cd')\}xy^2 + (ce')y^3 = 0.$$

Lastly, multiply the first by $a'x^3 + b'x^2y + c'xy^2 + d'y^3$; the second by $ax^3 + bx^2y + cxy^2 + dy^3$; subtract, and divide by y^4; when we get

$$(ae')x^3 + (be')x^2y + (ce')xy^2 + (de')y^3 = 0.$$

From the four equations thus formed we can eliminate linearly the four quantities, x^3, x^2y, xy^2, y^3, and obtain for our result the determinant

$$\begin{vmatrix} (ab'), & (ac') & , (ad') & , (ae') \\ (ac'), & (ad') + (bc'), & (ae') + (bd') & , (be') \\ (ad'), & (ae') + (bd'), & + (be') + (cd'), & (ce') \\ (ae'), & (be'), & (ce'), & (de') \end{vmatrix}$$

50. The process here employed is so evidently applicable to any two equations, both of the n^{th} degree, that it is unnecessary to make a formal statement of the general proof. On inspection of the determinant obtained in the last article, the law of its formation is apparent, and we can at once write down the determinant which is the eliminant between two equations of the fifth degree by simply continuing the series of terms, writing an (af') after every (ae'), &c. Thus the eliminant is

$$\begin{vmatrix} (ab'), & (ac') & ,(ad') & ,(ae') & ,(af') \\ (ac'), & (ad')+(bc'), & (ae')+(bd') & ,(af')+(be') & ,(bf') \\ (ad'), & (ae')+(bd'), & (af')+(be')+(cd'), & +(bf')+(ce') & ,(cf') \\ (ae'), & (af')+(be'), & +(bf')+(ce'), & +(cf')+(de'), & (df') \\ (af'), & +(bf'), & +(cf'), & +(df'), & (ef') \end{vmatrix}$$

It appears hence that in an eliminant every term must contain a or a'; as was evident beforehand, since if both of these were $= 0$, the equations would evidently have the common factor $y = 0$.

It appears also that those terms which contain a or a' only in the first degree are (ab') multiplied by the eliminant of the equations got by making a and $a' = 0$ in the given equations. For every element in the determinant written above must contain a constituent from the first row, and also one from the first column; but as all the constituents of the first row or column contain a or a', the only terms which contain a and a' in only the first degree, are (ab') multiplied by the corresponding minor; and this, when a and a' are made $= 0$, is the next lower eliminant.

51. It only remains to show that the process here employed is applicable when the equations are of different dimensions, and, as before, we commence with a particular example, viz., the equations

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0, \quad a'x^2 + b'xy + c'y^2 = 0.$$

Multiply the first by a', the second by ax^2, and subtract, when we have

$$(ba')x^3 + (ca')x^2y + (da')xy^2 + (ea')y^3 = 0.$$

In like manner, multiply the first by $a'x + b'y$, and the second by $(ax + by)\, x^2$, and we get

$$(ca')\, x^3 + \{(cb') + (da')\}\, x^2y + \{(db') + (ea')\}\, xy^2 + (eb')\, y^3 = 0.$$

This process can be carried no further; but if we join to the two equations just obtained the two equations got by multiplying the second of the original equations by x and by y, we have four equations from which to eliminate x^3, x^2y, xy^2, y^3.

And in general, when the degrees of the equations are unequal, m being the greater, it will be found that the process of Art. 49 gives us n equations of the $(m-1)^{st}$ degree, each of these equations being of the first order in the coefficients of *each* equation: to which we are to add the $m - n$ equations found by multiplying the second equation by x^{m-n-1}, $x^{m-n-2}y$, &c., and we can then eliminate the m quantities x^{m-1}, $x^{m-2}y$, &c., from the m equations we have formed. Every row of the determinant contains the coefficients of the second equation, but only n rows contain the coefficients of the first. The eliminant is, therefore, as it ought to be, of the n^{th} degree in the coefficients of the first, and of the m^{th} in those of the second equation.

52. Mr. Cayley has given a different statement of Bezout's method, explained in the last article. If two equations $\phi(x, y)$, $\psi(x, y)$, have a common root, then it must be possible to satisfy any equation of the form $\phi + \lambda\psi = 0$, independently of any particular value of λ. Take then the equation

$$\phi(x, y)\, \psi(x', y') - \phi(x', y')\, \psi(x, y) = 0;$$

which, if ϕ and ψ have a common factor, can be satisfied independently of any particular values of x' and y'. We may in the first place divide it by $xy' - yx'$, which is obviously a factor: then equate to 0 the coefficients of the several powers of x', y'; and then eliminate the powers of x and y as if they were independent variables, when the result comes out in precisely the same form as by the method of Art. 50.

Ex. To eliminate between $ax^2 + bxy + cy^2 = 0$, $a'x^2 + b'xy + c'y^2 = 0$,

$$(ax^2 + bxy + cy^2)\,(a'x'^2 + b'x'y' + c'y'^2) - (a'x^2 + b'xy + c'y^2)\,(ax'^2 + bx'y' + cy'^2),$$

when divided by $xy' - yx'$, gives

$$\{(ab')x + (ac')y\}\,x' + \{(ac')x + (bc')\,y\}\,y' = 0;$$

and equating to 0 the coefficients of x' and y', we get the eliminant

$$(ac')^2 + (ba')\,(bc') = 0.$$

53. We proceed now to the theory of functions of three variables, the eliminant of which, however, except in particular cases, has not been expressed as a determinant, though it can always be expressed as the quotient of one determinant divided by another.

We shall show, in the first place, how to form a function of great importance in the theory of elimination. Given n equations in n variables, $u = 0$, $v = 0$, $w = 0$, then the determinant

$$\begin{vmatrix} \dfrac{du}{dx}, & \dfrac{du}{dy}, & \dfrac{du}{dz} \\ \dfrac{dv}{dx}, & \dfrac{dv}{dy}, & \dfrac{dv}{dz} \\ \dfrac{dw}{dx}, & \dfrac{dw}{dy}, & \dfrac{dw}{dz} \end{vmatrix}$$

is called Jacobi's determinant, or simply the Jacobian of the given equations, and will be denoted in what follows by the letter J.

54. *If any number of equations are satisfied by a common system of values, that system will satisfy the Jacobian, and when the equations are of the same degree, it will also satisfy the differentials of the Jacobian with regard to each of the variables.*

The proof of this for three variables applies in general. For brevity we write $\dfrac{du}{dx} = a_1$, $\dfrac{du}{dy} = b_1$, $\dfrac{du}{dz} = c_1$, &c. : then, by the theorem of homogeneous functions, we have

$$a_1x + b_1y + c_1z = nu$$
$$a_2x + b_2y + c_2z = nv$$
$$a_3x + b_3y + c_3z = nw;$$

and, solving these equations, we have (Art. 24) $Jx = A_1nu + A_2nv + A_3nw$, from which it appears at once that if u, v, w vanish, J will vanish too. Again, differentiating the equation just found, we have

$$J + x\frac{dJ}{dx} = nu\frac{dA_1}{dx} + nv\frac{dA_2}{dx} + nw\frac{dA_3}{dx} + n\,(a_1A_1 + a_2A_2 + a_3A_3)$$

$$x\frac{dJ}{dy} = nu\frac{dA_1}{dy} + nv\frac{dA_2}{dy} + nw\frac{dA_3}{dy} + n\,(b_1A_1 + b_2A_2 + b_3A_3).$$

But, remembering (Arts. 22, 23,) that $a_1A_1 + a_2A_2 + a_3A_3 = J$; $b_1A_1 + b_2A_2 + b_3A_3 = 0$; we see that the supposition $u = 0$, $v = 0$, $w = 0$ (in consequence of which J is also $= 0$) makes $\frac{dJ}{dx}$, $\frac{dJ}{dy}$ also to vanish.

55. We can now express as a determinant the eliminant of three equations, each of the second degree. For their Jacobian is of the third degree, and therefore its differentials are of the second. We have thus three new equations of the second degree, which will be also satisfied by any system of values common to the given equations. From the six equations, then, u, v, w, $\frac{dJ}{dx}$, $\frac{dJ}{dy}$, $\frac{dJ}{dz}$, we can eliminate the six quantities x^2, y^2, z^2, yx, zx, xy, and so form the determinant required.

Again, if the equations are all of the third degree, J is of the sixth, and its differentials of the fifth, and if we multiply each of the three given equations by x^2, y^2, z^2, yz, zx, xy, we obtain 18 equations, which, combined with the three differentials of the Jacobian, enable us to eliminate dialytically the 21 quanities, x^5, x^4y, &c., which enter into an equation of the fifth degree. This process, however, cannot, without modification, be extended further.

56.* Mr. Sylvester has shown that the eliminant can always be expressed as a determinant when the three equations are of the same degree. Let us take, for an example, three equations of the fourth degree. Multiply each by the six terms (x^2, xy, y^2, &c.) of an equation of the second degree [or generally by the $\frac{n(n-1)}{2}$ terms of an equation of the $(n-2)^{nd}$ degree]. We thus form 18, $\left[\frac{3n(n-1)}{1 \,.\, 2}\right]$ equations. But since these equations, being now of the sixth $[2n-2]$ degree, consist of 28, $[n(2n-1)]$ terms; we require 10, $\left[\frac{n(n+1)}{2}\right]$ additional equations to enable us to eliminate dialytically all the powers of the variables. These

* The beginner may omit the rest of this Lesson.

equations are formed as follows. The first of the three given equations can be written in the form $Ax^4 + By + Cz$; the second and third, in the form $A'x^4 + B'y + C'z$, $A''x^4 + B''y + C''z$; and the determinant $(AB'C'')$ which is of the sixth degree in the variables must obviously be satisfied by any values which satisfy all the given equations. We should form two similar determinants by decomposing the equations into the form $Ay^4 + Bx + Cz$, $Az^4 + Bx + Cy$. So again, we might decompose the equations into the forms $Ax^3 + By^2 + Cz$, $A'x^3 + B'y^2 + C'z$, $A''x^3 + B''y^2 + C''z$ (for every term not divisible by x^3 or y^2 must be divisible by z); and then we obtain another determinant $(AB'C'')$ which will be satisfied when the equations vanish together. There are six determinants of this form got by interchanging x, y, and z in the rule for decomposing the equations. Lastly, decomposing into the form $Ax^2 + By^2 + Cz^2$ &c. we get a single determinant, which, added to the nine equations already found, makes the ten required. In general, we decompose the equations into the form $Ax^\alpha + By^\beta + Cz^\gamma$, such that $\alpha + \beta + \gamma = n + 2$, and form the determinant $AB'C''$; and it can be very easily proved that the number of integer solutions of the equation $\alpha + \beta + \gamma = n + 2$ is $\frac{n(n+1)}{2}$, exactly the number required.

57. When the degrees of the equations are different, it is not possible to form a determinant in this way, which shall give the eliminant clear of extraneous factors. The reason why such factors are introduced, and the method by which they are to be got rid of, will be understood from the following theory, due to Mr. Cayley. Let us take for simplicity three equations, u, v, w, all of the second degree. If we attempt to eliminate dialytically by multiplying each by x, y, z, we get nine equations, which are not sufficient to eliminate the ten quantities x^3, x^2y, &c. Again, if we multiply each equation by the six quantities, x^2, xy, y^2, &c., we have 18 equations, which are *more than* sufficient to eliminate the fifteen quantities x^4, x^3y, &c. If we take at pleasure any 15 of these equations, and form their determinant, we shall indeed have the eliminant, but it will be multiplied by an extraneous factor; since the determinant is of the

fifteenth degree in the coefficients, while the eliminant is only of the twelfth (Art. 38, $mn + np + pm = 12$, when $m = n = p = 2$). The reason of this is, that the 18 equations we have formed are not independent, but are connected by three linear relations. In fact, if we write the identity $uv = vu$, and then replace the first u by its value, $ax^2 + by^2 + \&c.$, and in like manner, with the v on the right-hand side of the equation, we get

$$ax^2v + by^2v + cz^2v + dyzv + \&c. = a'x^2u + b'y^2u + \&c.$$

In like manner, from the identities $vw = wv$, $wu = uw$, we get two other identical relations connecting the quantities x^2u, y^2u, x^2v, x^2w, &c. The question then comes to this: "If there be $m + p$ linear equations in m variables, but these equations connected by p linear relations so as to be equivalent only to m independent equations, how to express most simply the condition that all the equations can be made to vanish together." In the present case $m = 15$, $p = 3$.

58. Let us, for simplicity, take an example with numbers not quite so large, for instance, $m = 3$, $p = 1$. That is to say, let us consider four equations, s, t, u, v, where $s = a_1x + b_1y + c_1z$, $t = a_2x + b_2y + c_2z$, &c., these equations not being independent, but satisfying the relation, $D_1s + D_2t + D_3u + D_4v = 0$. Now I say, in the first place, that if we form the determinant $(a_1b_2c_3)$ of any three of these equations, s, t, u, this must contain D_4 as a factor. For if $D_4 = 0$, we shall have s, t, u connected by a linear relation, so that any values which satisfied both s and t should satisfy u also; and therefore the supposition $D_4 = 0$ would cause the determinant $(a_1b_2c_3)$ to vanish. And in the second place, I say that we get the same result (or, at least, one differing only in sign) whether we divide $(a_1b_2c_3)$ by D_4 or $(a_1b_2c_4)$ by D_3. For (Art. 14) $D_4(a_1b_2c_4)$ is the same as the determinant whose first row is a_1, b_1, c_1, the second, a_2, b_2, c_2, and the third, D_4a_4, D_4b_4, D_4c_4: but we may substitute for D_4a_4 its value $-D_1a_1 - D_2a_2 - D_3a_3$, and in like manner for D_4b_4, D_4c_4. The determinant would then (Art. 16) be resolvable into the sum of three others; but two of these would vanish, having two rows the same, and there would remain $D_4(a_1b_2c_4) = -D_3(a_1b_2c_3)$. It follows then that the

eliminant of the system may be expressed in any of the equivalent forms obtained by forming the determinant $(a_1b_2c_3)$ of any three of the equations, and dividing by the remaining constant D_4.

Suppose now that we had five equations s, t, u, v, w, connected by two linear relations $D_1s + D_2t + D_3u + D_4v + D_5w = 0$, $E_1s + E_2t + E_3u + E_4v + E_5w = 0$. Eliminating w from these relations, we have $(D_1E_5)s + (D_2E_5)t + (D_3E_5)u + (D_4E_5)v = 0$, and we see, precisely as before, that the supposition $(D_4E_5) = 0$ would cause the determinant $(a_1b_2c_3)$ to vanish; and that we get the same result whether we divide $(a_1b_2c_3)$ by (D_4E_5) or divide the determinant of any other three of the equations by the complemental determinant answering to (D_4E_5). This reasoning may be extended to any number of equations connected by any number of relations, and we are led to the following general rule for finding the eliminant of the system in its simplest form. Write down the constants in the $m + p$ equations, and complete them into a square form by adding the constants in the p relations: thus—

$$\begin{array}{llll|ll} s; & a_1, & b_1, & c_1 & D_1, & E_1 \\ t; & a_2, & b_2, & c_2 & D_2, & E_2 \\ u; & a_3, & b_3, & c_3 & D_3, & E_3 \\ v; & a_4, & b_4, & c_4 & D_4, & E_4 \\ w; & a_5, & b_5, & c_5 & D_5, & E_5 \end{array}$$

then the eliminant in its most reduced form is the determinant of any m rows of the left-hand or equation columns, divided by the determinant got by erasing these rows in the right-hand columns.

Thus, then, in the example of the last Article, we take the determinant of any 15 of the equations, and, dividing it by a determinant formed with three of the relation rows, obtain the eliminant; which is of the twelfth degree, as it ought to be.

59. And, in general, given three equations of the m^{th}, n^{th}, and p^{th} degrees, we form a number of equations of the degree $m + n + p - 2$, by multiplying the first equation by all the terms x^{n+p-2},

$x^{n+p-3}y$, and so on. We should in this manner have $\frac{(n+p-1)(n+p)}{2}$ $+ \frac{(p+m-1)(p+m)}{2} + \frac{(m+n-1)(m+n)}{2}$ equations. But the number of terms, $x^{m+n+p-2}$, &c., to be eliminated from the equations formed, is $\frac{(m+n+p-1)(m+n+p)}{2}$, or, in general, less than the number of equations. But again, if we consider the identity $uv = vu$, which is of the degree $m+n$, and multiply it by the several terms x^{p-2}, &c., we get $\frac{(p-1)p}{2}$ identical relations between the system of equations we have formed; and in like manner $\frac{(n-1)n}{2} + \frac{(m-1)m}{2}$ other identities; and the number of identities subtracted from the number of equations leaves exactly the number of variables to be eliminated, and gives the eliminant in the right degree.

60. If we had four equations in four variables, we should proceed in like manner, and it would be found then that the case would arise of our having $m+n$ linear equations in m variables, these equations not being independent, but connected by $n+p$ relations; these latter relations again not being independent, but connected by p other relations. And in order to find the reduced eliminant of such a system, we should divide the determinant of any m of the equations by a quantity which is itself the quotient of two determinants. I think it needless to go into further details, but I thought it necessary to explain so much of the theory, the above being, as far as I know, the only general theory of the expression of eliminants as determinants; since whenever, in the application of the dialytic method, any of the equations is multiplied by terms exceeding its own degree, we shall be sure to have a number of equations greater than the number of quantities which we want to eliminate.

LESSON VII.

DISCRIMINANTS.

61. Before entering on the subject of discriminants, we shall explain some terms and symbols which we shall frequently find it convenient to employ. In ordinary algebra we are wholly concerned with *equations*, the object usually being to find the values of x which will make a given function $= 0$. In what follows we have little to do with equations, the most frequent subject of investigation being that on which we enter in the next Lesson: namely, the discovery of those properties of a function which are unaltered by linear transformations. It is convenient, then, to have a word to denote the function itself, without being obliged to speak of the equation got by putting the function $= 0$: a word, for example, to denote $ax^2 + bxy + cy^2$ without being obliged to speak of the quadratic equation $ax^2 + bxy + cy^2 = 0$. We shall, after Mr. Cayley, use the term *quantic* to denote a homogeneous function in general; using the words quadric, cubic, quartic, quintic, n^{ic}, to denote quantics of the 2nd, 3rd, 4th, 5th, n^{th} degrees. And we distinguish quantics into binary, ternary, quaternary, n-ary, according as they contain 2, 3, 4, n variables. Thus, by a binary cubic, we mean a function such as $ax^3 + bx^2y + cxy^2 + dy^3$; by a ternary quadric, such as $ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy$ &c. Mr. Cayley uses the abbreviation $(a, b, c, d \between x, y)^3$ to denote the quantic $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, in which, as is usually most convenient, the terms are affected with the same numerical coefficients as in the expansion of $(x + y)^3$. So the ternary quadric written above would be expressed $(a, b, c, d, e, f \between x, y, z)^2$. When the terms are not thus affected with numerical coefficients, he puts an arrow-head on the parenthesis, writing, for instance $(a, b, c, d \between x, y)^3$ to denote $ax^3 + bx^2y + cxy^2 + dy^3$. When it is not necessary to mention the coefficients, the quantic of the n^{th} degree is written $(x, y)^n$, $(x, y, z)^n$, &c.

62. If a quantic in p variables be differentiated with respect to each of the variables, the eliminant of the p differentials is called the *discriminant* of the given quantic.

If n be the degree of the quantic, its discriminant is a homogeneous function of its coefficients, and of the degree $p(n-1)^{p-1}$. For the discriminant is the eliminant of p equations of the $(n-1)^{st}$ degree, and (Art. 40) must contain the coefficients of *each* of these equations in a degree equal to the product of the degrees of all the rest, that is $(n-1)^{p-1}$. And since each of these equations contains the coefficients of the original quantic in the first degree, the discriminant contains them in the $p(n-1)^{p-1}$ degree. Thus, then, the discriminant of a binary quantic is of the degree $2(n-1)$; of a ternary, is of the degree $3(n-1)^2$; &c.

63. *If in the original quantic every coefficient multiplying the first power of one of the variables x, be affected with a suffix 1, every term multiplying the second power by a suffix 2, and so on; then the sum of the suffixes in each term of the discriminant is constant and $= n(n-1)^{p-1}$.* It was proved (Art. 40) that if every coefficient in a system of equations were affected with a suffix corresponding to the power of x which it multiplies, then the sum of the suffixes in every term of their eliminant would be equal to the product of the degrees of those equations, viz., $= mnp$ &c. Now suppose, that in the first of these equations the suffix of x^0, instead of being 0, was k; that of x^1 was $k+1$, and so on; it is evident that the effect would be to increase the sum of the suffixes by k for every coefficient of the first equation which enters into the eliminant; and since (Art. 40) every term contains np &c., coefficients of the first equation; the total sum of suffixes is mnp &c. $+ knp$ &c. $= (m+k)np$ &c.* Now, in the present example, it is evident that every coefficient in the $p-1$ differentials $\frac{dU}{dy}, \frac{dU}{dz}$, &c., multiplies the same power of x

* In like manner, if the suffixes in the second equation were all increased by k', those in the third by k'', &c., then the sum of suffixes in the eliminant would be $(m+k)(n+k')(p+k'')$ &c.

as it did in the original quantic U. But in the remaining differential, $\frac{dU}{dx}$, every coefficient multiplies a power of x one less than in U, and the coefficient multiplying any term x^k in this differential will be marked with the suffix $k+1$, since it arose from differentiating a term x^{k+1} in the original quantic. It follows, then, that the sum of suffixes in the discriminant must $= (n-1)^p + (n-1)^{p-1} = n(n-1)^{p-1}$.

We shall briefly express the results of this and of the last article by saying that the *order* of the discriminant is $p(n-1)^{p-1}$; and its *weight*, $n(n-1)^{p-1}$. Thus for a binary quantic the weight of the discriminant is $n(n-1)$.

64. If a binary quantic contain a square factor, then, as is well known, the discriminant vanishes identically. For the two differentials must each contain that factor in the first degree, and therefore, since they have a common factor, their eliminant vanishes. In like manner, if a ternary quantic be of the form $X^2\phi + XY\psi + Y^2\chi$, where $X = ax + by + cz$, $Y = a'x + b'y + c'z$, then the discriminant must vanish, since every term in any of the differentials must contain either X or Y, and therefore the differentials have common the system of roots derived from the equations $X = 0$, $Y = 0$. In like manner, the discriminant of a quaternary quantic vanishes, if the quantic can be expressed as a function in the second degree of X, Y, Z, these being any linear functions of the variables.* We shall call those values which make all the differentials vanish, the *singular roots* of the quantic.

65. We shall now discuss the properties of the discriminant of the binary quantic $U = a_0x^n + na_1x^{n-1}y + \frac{n(n-1)}{1 \cdot 2} a_2x^{n-2}y^2 + \&c.$

The eliminant of U *and* $\frac{dU}{dx}$ *is* a_0 *times the discriminant; and*

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* In other words, the vanishing of the discriminant of an algebraic equation expresses the condition that the equation shall have equal roots; and the vanishing of the discriminant of the equation of a curve or surface expresses the condition that the curve or surface shall have a double point.

*the eliminant of U and $\frac{dU}{dy}$ is a_n times the discriminant.** For since $nU = x\frac{dU}{dx} + y\frac{dU}{dy}$, the result of substituting in nU any root of $\frac{dU}{dx}$ is $y'\left(\frac{dU}{dy}\right)'$; and when all the results of substitution are multiplied together, the product will be $y'y''y'''$ &c. (which is $= a_0$, see note, p. 21), multiplied by the product of the results of substituting the same roots in $\frac{dU}{dy}$, which is the discriminant.

66. *To express the discriminant in terms of the values x_1y_1 x_2y_2 &c., which make the quantic vanish.*

Let $U = (xy_1 - yx_1)(xy_2 - yx_2)(xy_3 - yx_3)$ &c. (see note, p. 21); then $\frac{dU}{dx} = y_1(xy_2 - yx_2)(xy_3 - yx_3)$&c.$+ y_2(xy_1 - yx_1)(xy_3 - yx_3)$&c.+&c.; and the result of the substitution in $\frac{dU}{dx}$ of any root x_1y_1 of U is $y_1(x_1y_2 - y_1x_2)(x_1y_3 - y_1x_3)$ &c. Similarly, the result of substituting x_2y_2 is $y_2(x_2y_1 - x_1y_2)(x_2y_3 - y_2x_3)$ &c. If, then, all the results of substitution are multiplied together, the product is

$$\pm\, y_1y_2y_3 \text{ \&c. } (x_1y_2 - y_1x_2)^2(x_1y_3 - y_1x_3)^2(x_2y_3 - y_2x_3)^2 \text{ \&c.}$$

This, then is the eliminant of U and $\frac{dU}{dx}$, and if we divide it by a_0, which is $= y_1y_2y_3$ &c., we shall have the discriminant $= (x_1y_2 - y_1x_2)^2(x_1y_3 - y_1x_3)^2$ &c. If we make in it all the y's $= 1$, we get the theorem in the well-known form that the discriminant is equal to the product of the squares of all the differences of any two roots of the equation. We shall, for simplicity, refer to the theorem in the latter form.

67. *The discriminant of the product of two quantics is equal to the product of their discriminants multiplied by the square of their eliminant.* For the product of the squares of differences of all the roots evidently consists of the product of the squares of differences of two roots both belonging to the same quantic, multiplied by the square of the product of all differences between a root of

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* We do not take account of mere numerical factors.

one and a root of the other, and this latter product is the eliminant (Art. 31). As a particular case of this, the discriminant of $(x-a)\,\phi(x)$ is the discriminant of $\phi(x)$ multiplied by the square of $\phi(a)$. For if β, γ, &c., be the roots of $\phi(x)$, then $(a-\beta)^2(a-\gamma)^2$ $(\beta-\gamma)^2$ &c. is equal to the square of $(a-\beta)(a-\gamma)$ &c., which is $\phi(a)$, multiplied by the product of the squares of all differences not containing a.

68. *The discriminant of* $(a_0, a_1 \ldots a_{n-1}, a_n \between x, y)^n$ *is of the form* $a_n\phi + a^2_{n-1}\psi$, *where* ψ *is the discriminant of the equation of the* $(n-1)^{st}$ *degree* $(a_0, a_1 \ldots\ldots a_{n-2}, a_{n-1} \between x, y)^{n-1}$. For we evidently get the same result whether we put any term $a_n = 0$ in the discriminant; or whether we put $a_n = 0$ in the quantic, and then form the discriminant. But if we make $a_n = 0$ in the quantic, we get x multiplied by the $(n-1)^{ic}$ written above, and (Art. 67) its discriminant will then be the discriminant of that $(n-1)^{ic}$ multiplied by the square of the result of making in it $x = 0$; that is, by the square of a_{n-1}. In like manner we see that the discriminant is of the form $a_0\phi + a_1{}^2\psi$.

69. If the discriminant of a binary quantic vanishes, then the equal roots are found from the theorem, that *the several differential coefficients of the discriminant with respect to* a_0, a_1, *&c., are proportional to the differential coefficients of the quantic with respect to the same quantities*. For, let $U = a_0x^n + a_1x^{n-1} + a_2x^{n-2} +$ &c., and let a be one of its equal roots, then a will still be a root of the quantic $(a_0 + \lambda A_0)\,x^n + (a_1 + \lambda A_1)\,x^{n-1} +$ &c., provided only that A_0, A_1, &c., are connected by the equation $V = A_0a^n + A_1a^{n-1} + A_2a^{n-2} + \text{\&c.} = 0$. But now I say that the discriminant of $U + \lambda V$ must be divisible by λ^2; for if $U + \lambda V = (x-a)$ $\{\phi(x) + \lambda\psi(x)\}$; then (Art. 67) the discriminant of $U + \lambda V$ will be divisible by the square of $\{\phi(a) + \lambda\psi a\}$; or, since $\phi(a) = 0$, will be divisible by λ^2. But if Δ be the discriminant of U, the discriminant of $U + \lambda V$ is formed from it by altering in it a_0 into $a_0 + \lambda A_0$, a_1 into $a_1 + \lambda A_1$ &c., and is therefore

$$= \Delta + \lambda\left(A_0\frac{d\Delta}{da_0} + A_1\frac{d\Delta}{da_1} + A_2\frac{d\Delta}{da_2} + \text{\&c.}\right) + \lambda^2\,(\text{\&c.}).$$

By hypothesis $\Delta = 0$, and we have now proved that the coeffi-

cient of λ must also vanish; but this relation must be the same as that which we already know to be the only one which need connect A_0, A_1, &c. By comparing, therefore, any two coefficients, we find the value of a. This proof holds equally if a_0, a_1, instead of being the coefficients themselves, were any quantities of which the coefficients were functions.*

70. We shall give another proof of the theorem of the last article which will be applicable to the case of a quantic in any number of variables. For simplicity, we confine ourselves to the case of three variables. It was proved (Art. 45) that if a and b be any two constants which enter into three quantics ϕ, ψ, χ, then the differential coefficients of the eliminant with respect to these constants are proportional to certain determinants. But when ϕ, ψ, χ are the three differentials of the

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* These results may be confirmed by forming the actual values of $\frac{d\Delta}{da_1}$ &c. in terms of the roots; which I do by solving from the n equations

$$\frac{d\Delta}{da} = \frac{d\Delta}{da_1}\frac{da_1}{da} + \frac{d\Delta}{da_2}\frac{da_2}{da} + \&c.$$

We know the expressions for Δ, a_1, a_2, &c. in terms of the roots, and therefore, from these n equations can find the n sought quantities $\frac{d\Delta}{da_1}$ &c. I find—

$$\frac{d\Delta}{da_n} = \Sigma(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2\{(\alpha-\beta)(\alpha-\gamma)+(\alpha-\beta)(\alpha-\delta)+(\alpha-\gamma)(\alpha-\delta)\}$$

where the product of the *squares* of all the differences not containing α is multiplied by the product of the differences between α and $n-2$ of the remaining roots:

$$\frac{d\Delta}{da_{n-1}} = \Sigma\alpha(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2\{(\alpha-\beta)(\alpha-\gamma)+\&c.\}$$

$$\frac{d\Delta}{da_{n-2}} = \Sigma\alpha^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2\{(\alpha-\beta)(\alpha-\gamma)+\&c.\}, \&c.,$$

and the supposition $\alpha=\beta$ reduces these sums to quantities which are in the ratio 1, α, α^2, &c. If more than two of the roots are equal to each other, all these differentials vanish identically, and we find the equal roots by proceeding to second differentials of the discriminant. A simpler series of symmetrical functions, however, can be formed possessing the same property, as suggested (Art. 47), namely,

$$\Sigma(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2;\ \Sigma\alpha(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2;\ \Sigma\alpha^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\beta)^2;$$

this series being of an order one lower in the coefficients.

same quantic, we have $\frac{d\psi}{dx} = \frac{d\phi}{dy}, \frac{d\chi}{dx} = \frac{d\phi}{dz}, \frac{d\chi}{dy} = \frac{d\psi}{dz}$, and we have $\frac{d\Delta}{da}$ and $\frac{d\Delta}{db}$ proportional to the two following determinants (it being understood that the x, y, z which occur in them are the singular roots):—

$$\begin{vmatrix} \frac{d\phi}{da}, & \frac{d\phi}{dx}, & \frac{d\psi}{dx} \\ \frac{d\psi}{da}, & \frac{d\phi}{dy}, & \frac{d\psi}{dy} \\ \frac{d\chi}{da}, & \frac{d\phi}{dz}, & \frac{d\psi}{dz} \end{vmatrix} : \begin{vmatrix} \frac{d\phi}{db}, & \frac{d\phi}{dx}, & \frac{d\psi}{dx} \\ \frac{d\psi}{db}, & \frac{d\phi}{dy}, & \frac{d\psi}{dy} \\ \frac{d\chi}{db}, & \frac{d\phi}{dz}, & \frac{d\psi}{dz} \end{vmatrix}.$$

But, since x, y, z satisfy ϕ and ψ, we have

$$x\frac{d\phi}{dx} + y\frac{d\phi}{dy} + z\frac{d\phi}{dz} = 0, \quad x\frac{d\psi}{dx} + y\frac{d\psi}{dy} + z\frac{d\psi}{dz} = 0.$$

Hence

$$x : y : z :: \frac{d\phi}{dy}\frac{d\psi}{dz} - \frac{d\phi}{dz}\frac{d\psi}{dy} : \frac{d\phi}{dz}\frac{d\psi}{dx} - \frac{d\phi}{dx}\frac{d\psi}{dz} : \frac{d\phi}{dx}\frac{d\psi}{dy} - \frac{d\phi}{dy}\frac{d\psi}{dx}.$$

Expanding then the two determinants above, their ratio is

$$x\frac{d\phi}{da} + y\frac{d\psi}{da} + z\frac{d\chi}{da} : x\frac{d\phi}{db} + y\frac{d\psi}{db} + z\frac{d\chi}{db};$$

or, since U is proportional to $x\phi + y\psi + z\chi$, we have for the sought x, y, z,

$$\frac{d\Delta}{da} : \frac{d\Delta}{db} :: \frac{dU}{da} : \frac{dU}{db}. \quad \text{Q. E. D.}$$

71. We may now, by reversing the process of Art. 69, extend to any quantic the theorem proved for binary quantics (Art. 68):—"That if a be the coefficient of the highest power of one of these variables; b, c, d, &c., those of the next highest power; then the discriminant is of the form

$$a\theta + (\phi, \chi, \psi, \&\text{c.})(b, c, d, \&\text{c.})^2."$$

Thus, for a ternary quantic, if a be the coefficient of x^n; b, c those of $x^{n-1}y$, $x^{n-1}z$, then, if in the discriminant we make $a = 0$,

the remaining part will be of the form $b^2\phi + bc\psi + c^2\chi$.*
To prove this: first, let U be any quantic whose discriminant vanishes, V any other satisfied by the singular roots of U, then I say that the discriminant of $(U + V\lambda)$ will be divisible by λ^2. For, let $U = ax^n + bx^{n-1}y + cx^{n-1}z + \&c.$, $V = Ax^n + Bx^{n-1}y + Cx^{n-1}z + \&c.$, then the coefficient of λ in the discriminant of $U + \lambda V$ will be $A\frac{d\Delta}{da} + B\frac{d\Delta}{db} + C\frac{d\Delta}{dc} + \&c.$: but it has been proved, Art. 70, that $\frac{d\Delta}{da}, \frac{d\Delta}{db}$, &c., are respectively proportional to x^n, $x^{n-1}y$, &c.; the coefficient of λ is therefore proportional to the result of substituting the singular roots in V, which, by hypothesis, $= 0$.

Now, confining ourselves for simplicity to the case of a ternary quantic, if a be the coefficient of the highest power of z, b and c those of $z^{n-1}x$, $z^{n-1}y$, then the supposition of $a = 0$, $b = 0$, $c = 0$, must make the discriminant vanish, since then all the differentials vanish for the singular roots $x = 0$, $y = 0$. In order that any other quantic V should vanish for the same values $x = 0$, $y = 0$, it is only necessary that $A = 0$. The general form of the discriminant then must be such that if we substitute for b, $b + \lambda B$, for c, $c + \lambda C$, and then make $a, b, c = 0$, the result must be divisible by λ^2; or, in other words, if we put for b, λB, and for c, λC, and then make $a = 0$, the result is divisible by λ^2, which was the thing to be proved.

LESSON VIII.

INVARIANTS.

72. We have seen (Art. 18) that the fundamental theorem of the multiplication of determinants may be stated as follows:— Let there be any number of equations of the first degree, and let

* This theorem has some geometrical applications.

the variables in each be transformed by the same substitutions, viz.: for x, $a_1x+\beta_1y+\gamma_1z+$ &c.; for y, $a_2x+\beta_2y+\gamma_2z+$ &c.; for z, $a_3x+\beta_3y+\gamma_3z+$ &c.; &c.; then the determinant is a function of the coefficients of the original equations, such that the same function of the coefficients of the transformed equations is equal to the original determinant multiplied by the modulus of transformation $(a_1\beta_2\gamma_3)$.

Now, in general, let there be an equation, or system of equations, *of any degree*, then any function of their coefficients is called an *invariant*, provided it possesses this same property, viz., that if the equations be transformed by linear substitutions for each of the variables, such as those given above, then the same function of the new coefficients is equal to the old function multiplied by some power of the modulus of transformation. We shall, in general, suppose the transformation to be *uni-modular*, that is to say, such that $(a_1\beta_2\gamma_3)$ the modulus of transformation is equal to unity, and then an invariant is a function of the coefficients which is absolutely unaltered by linear transformation.

Let us take, as the simplest example, the quadric $ax^2+2bxy+cy^2$, then its discriminant $ac-b^2$ is an invariant. For, let the variables be transformed, and the quantic becomes

$$a(a_1x+\beta_1y)^2+2b(a_1x+\beta_1y)(a_2x+\beta_2y)+c(a_2x+\beta_2y)^2,$$

and if the transformed quantic be $Ax^2+2Bxy+Cy^2$, then we have

$$A=aa_1^2+2ba_1a_2+ca_2^2,\quad C=a\beta_1^2+2b\beta_1\beta_2+c\beta_2^2,$$
$$B=aa_1\beta_1+b(a_1\beta_2+a_2\beta_1)+ca_2\beta_2,$$

and it can be verified without difficulty that

$$AC-B^2=(ac-b^2)(a_1\beta_2-a_2\beta_1)^2.$$

Again, to take an example of a system of equations, take the two quadrics, $ax^2+2bxy+cy^2$, $a'x^2+2b'xy+c'y^2$; then it can be verified by actual expansion and multiplication, that if both equations be transformed by linear substitutions,

$$AC'+CA'-2BB'=(ac'+ca'-2bb')(a_1\beta_2-a_2\beta_1)^2.$$

The function $ac'+ca'-2bb'$ is therefore an invariant of this system of equations.

73. Having explained the meaning of the word *invariant* we proceed next to explain the meaning of the word *covariant*. Given any quantic, and another derived from it according to any rule, the derived quantic is said to be a covariant, provided that when the variables in both are transformed by the same linear substitutions, then the result obtained by transforming the derived quantic is the same as that obtained by forming the similar derived of the transformed quantic (or at least differs from it only by a power of the modulus of transformation). Thus, take the quantic U, and the derived $\dfrac{dU}{dx}$, this derived is *not* a covariant, because, if we transform U, and then differentiate with regard to x, we do not get the same result as if we were first to differentiate, and then transform. But it will be presently shown in various ways that the derivative $\dfrac{d^2U}{dx^2}\dfrac{d^2U}{dy^2}-\left(\dfrac{d^2U}{dxdy}\right)^2$ *is* a covariant, that is to say, that it will be transformed into $\dfrac{d^2V}{dx^2}\dfrac{d^2V}{dy^2}-\left(\dfrac{d^2V}{dxdy}\right)^2$, where V is the transformed of U.

In this and the following Lessons our object is to point out ways by which the invariants and covariants of a given quantic can be found; and we confine ourselves in this Lesson to the theory of binary quantics.*

74. *The discriminant of a binary quantic, or the eliminant of a system of binary quantics, is an invariant.*

We can see *a priori* that this must be the case, for if a given quantic has a square factor, it will have a square factor still when it is linearly transformed; or if a system of quantics have a com-

* In the geometry of curves and surfaces, all transformations of co-ordinates are effected by linear transformations; invariants, then, are functions of the coefficients expressing certain fixed properties of the curve or surface which are independent of our choice of axes; such as the condition that a curve or surface should have a double point, &c. Covariants represent certain other curves or surfaces having a fixed relation to the given one, independent of our choice of axes. The reader will see then that the theory of the discovery of the invariants or covariants of a quantic must lead to the solution of many important geometrical problems.

mon factor, they will still have a common factor when the equations are transformed. The discriminant or eliminant of the transformed equations must, therefore, always vanish whenever the discriminant or eliminant of the original system vanishes; the one must, therefore, contain the other as a factor.* The theorem, however, can be formally proved from the expression for the discriminant in terms of the roots. Let x_1, y_1 be values which satisfy U, then a factor of U is $xy_1 - yx_1$, but when the variables are transformed, this becomes $(a_1x+\beta_1y)\,y_1-(a_2x+\beta_2y)\,x_1$. We see then that if we write the corresponding factor of the transformed U, $xY_1 - yX_1$, we must have $X_1 = -(\beta_1y_1 - \beta_2x_1)$, $Y_1 = a_1y_1 - a_2x_1$. Now the expression for the discriminant or eliminant (Arts. 31, 66) is the product of a number of terms of the form $x_1y_2 - y_1x_2$. But from the above values for X_1, Y_1, and the corresponding expressions for X_2, Y_2, we can at once prove that $(X_1Y_2 - Y_1X_2) = (x_1y_2 - y_1x_2)(a_1\beta_2 - a_2\beta_1)$. It follows then that the discriminant or eliminant of the transformed system is equal to the original multiplied by a power of $a_1\beta_2 - a_2\beta_1$ equal to the number of factors in the expression for the discriminant or eliminant in terms of the roots.

75. It is evident, in like manner, that any symmetric function of the roots of a quantic which can be expressed as the product of a number of factors $x_1y_2 - y_1x_2$, will be an invariant. Now, when we are expressing any ordinary algebraic symmetric function of the roots, in the language of homogeneous equations, we substitute for each root $\frac{x_1}{y_1}$, and then clear of fractions by multiplying by a power of the coefficient of x^n, which is the product of all the y's, $y_1y_2y_3y_4$ &c. Suppose now we were given in a quartic $\Sigma(a-\beta)^2$, this would become $\Sigma\left(\frac{x_1}{y_1}-\frac{x_2}{y_2}\right)^2$, and, cleared of fractions, would be $\Sigma y_3^2y_4^2(x_1y_2-y_2x_1)^2$ which is *not* of the desired form. But any *symmetric function of the differences of the roots*

* The same reasoning shows that discriminants or eliminants of quantics in any number of variables are invariants.

is an invariant, provided that each root enters the same number of times into the expression. Thus, $\Sigma(\alpha-\beta)^2(\gamma-\delta)^2$ in like manner becomes $\Sigma(x_1y_2-y_1x_2)^2(x_3y_4-y_3x_4)^2$, and, therefore, is an invariant in the case of a quartic, but would not be so in the case of a sextic, for the expression would then become

$$\Sigma y_5^2 y_6^2 (x_1y_2-y_1x_2)^2 (x_3y_4-y_3x_4)^2.*$$

It is proved in like manner that any symmetric function formed of differences of roots and differences between x and one or more roots is a covariant, as, for instance, for a cubic, $\Sigma(x-\alpha)^2(\beta-\gamma)^2$, it being understood that each root enters the same number of times into the expression.

76. Let us next examine, when a quantic is linearly transformed; by what rules its differentials are transformed. Suppose that we write $x = a_1X + b_1Y$, $y = a_2X + b_2Y$, values which give conversely $X = \frac{1}{M}(b_2x - b_1y)$, $Y = \frac{1}{M}(-a_2x + a_1y)$, where for brevity we have written $M = a_1b_2 - a_2b_1$, the modulus of transformation. Suppose, then, that on making the above substitutions $\phi(x, y)$ becomes $\Phi(X, Y)$, we want to investigate the values of $\frac{d\phi}{dx}, \frac{d\phi}{dy}$ in terms of $\frac{d\Phi}{dX}, \frac{d\Phi}{dY}$. Now we have

$$\frac{d}{dx} = \frac{d}{dX}\frac{dX}{dx} + \frac{d}{dY}\frac{dY}{dx};\quad \frac{d}{dy} = \frac{d}{dX}\frac{dX}{dy} + \frac{d}{dY}\frac{dY}{dy},$$

or, from the values above given for X and Y,

$$\frac{d}{dx} = \frac{1}{M}\left(b_2\frac{d}{dX} - a_2\frac{d}{dY}\right);\quad \frac{d}{dy} = \frac{1}{M}\left(-b_1\frac{d}{dX} + a_1\frac{d}{dY}\right)$$

and all powers of $\frac{d}{dx}, \frac{d}{dy}$ are transformed by the same rule. But the above values may be written

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* All the invariants of binary quantics can be obtained in this way, which I gave ("Higher Plane Curves," p. 297); but the expression of symmetric functions in terms of the coefficients is tedious, and the method itself not capable of being extended to quantics in any number of variables. For this reason we go on to explain other methods of discovering invariants.

$$\frac{d}{dy} = \frac{1}{M}\left\{a_1 \frac{d}{dY} + b_1\left(-\frac{d}{dX}\right)\right\};\ -\frac{d}{dx} = \frac{1}{M}\left\{a_2\frac{d}{dY} + b_2\left(-\frac{d}{dX}\right)\right\}.$$

Thus it is seen that, with the exception of the constant divisor M, $\frac{d}{dy}$ and $-\frac{d}{dx}$ are transformed by exactly the same rules as x and y; and that if we have by transformation formed any equation $\phi(x, y) = \Phi(X, Y)$ we can, without calculation, write down $\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right) = \frac{1}{M^n}\Phi\left(\frac{d}{dY}, -\frac{d}{dX}\right)$; the index of M being equal to the degree of $\phi(x, y)$ in x and y.

77. To take an example, let us suppose that $ax^2 + 2bxy + cy^2$ becomes by transformation $= AX^2 + 2BXY + CY^2$; then we can write down

$$a\frac{d^2}{dy^2} - 2b\frac{d^2}{dxdy} + c\frac{d^2}{dx^2} = \frac{1}{M^2}\left(A\frac{d^2}{dY^2} - 2B\frac{d^2}{dXdY} + C\frac{d^2}{dX^2}\right).$$

It follows then, on applying this operative symbol to the given quantic, that

$$\left(a\frac{d^2}{dy^2} - 2b\frac{d^2}{dxdy} + c\frac{d^2}{dx^2}\right)(ax^2 + 2bxy + cy^2)$$

is equal to the corresponding function of the transformed equation divided by M^2, or by actual performance of the operation $ac - b^2 = \frac{1}{M^2}(AC - B^2)$, as was proved already (Art. 72).

In like manner, given a system of two quantics $ax^2 + 2bxy + cy^2$, $a'x^2 + 2b'xy + c'y^2$; if we operate on the second with

$$a\frac{d^2}{dy^2} - 2b\frac{d^2}{dxdy} + c\frac{d^2}{dx^2},$$

we get $ac' + ca' - 2bb'$, which we asserted was an invariant (Art. 72).

Again, from any binary quantic of even degree we can form in the same way an invariant of the second order in the coefficients. Thus operating on

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$$

with

$$a\frac{d^4}{dy^4} - 4b\frac{d^4}{dy^3dx} + 6c\frac{d^4}{dx^2dy^2} - 4d\frac{d^4}{dx^3dy} + e\frac{d^4}{dx^4},$$

we get $ae - 4bd + 3c^2$, which is therefore an invariant.

But if we treat a quantic of odd degree in like manner, we get no invariant, since the result will be found to vanish identically. Thus $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, operated on with

$$a\frac{d^3}{dy^3} - 3b\frac{d^3}{dxdy^2} + 3c\frac{d^3}{dx^2dy} - d\frac{d^3}{dx^3}$$

vanishes identically.

78. We proceed to define some new terms which it will often be convenient to employ. Two sets of variables $x, y, z; x', y', z'$, are said to be *cogredient* when it is understood that they are always to be transformed by the same linear substitutions, that is to say, that if we substitute for x, $a_1X + \beta_1Y + \gamma_1Z$, we are at the same time to substitute for x', $a_1X' + \beta_1Y' + \gamma_1Z'$, &c. Thus, for instance, if xyz are the running co-ordinates of a point on a curve, and $x'y'z'$ those of a fixed point which enter into the equation of the curve, then, if we transform x, y, z to any new axes of co-ordinates, we must of course make a corresponding change in $x'y'z'$.

Again, if in any quantic we substitute $x + \lambda x'$, $y + \lambda y'$, $z + \lambda z'$, &c., for x, y, z, where $x'y'z'$ are cogredient with xyz, then the coefficients of the several powers of λ, which are all of the form $\left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz} + \text{\&c.}\right)^p U$, are called the first, second, third, &c., *emanants* of the quantic.*

79. *Every emanant is a covariant to the original quantic.*

It is easy to see that we get the same result whether in any quantic we write $x + \lambda x'$ for x, &c., and then transform x and x' by linear substitutions, or whether we make the substitutions first, and then write $X + \lambda X'$ for X, &c. For, evidently,

* The geometrical reader is already familiar with emanants under the name of "polar curves or surfaces" of a point with regard to a given curve or surface.

$$\alpha_1 X + \beta_1 Y + \gamma_1 Z + \lambda(\alpha_1 X' + \beta_1 Y' + \gamma_1 Z') = \alpha_1(X + \lambda X') + \beta_1(Y + \lambda Y') + \gamma_1(Z + \lambda Z').$$

If, then, ϕ becomes by transformation Φ, so that $\phi(x, y) = \Phi(X, Y)$ we have proved that the result of writing $x + \lambda x'$, $y + \lambda y'$ &c., for x, y &c. in ϕ is the same as the result of writing $X + \lambda X'$ for X &c. in Φ; and as λ is indeterminate, the coefficients of the several powers of λ must be equal on both sides of the equation; or we have

$$x'\frac{d\phi}{dx} + y'\frac{d\phi}{dy} + \&c. = X'\frac{d\Phi}{dX} + Y'\frac{d\Phi}{dY} + \&c., \&c. \quad \text{Q. E. D.}$$

80. *If we consider any emanant as a function of x', y', &c. regarding x, y, &c. as constants; then any invariant of the emanant will be a covariant of the original quantic, when x, y, &c. are regarded as variables.*

The general proof will, perhaps, be better understood if we apply it first to a particular case. Take the second emanant

$$x'^2\frac{d^2U}{dx^2} + 2x'y'\frac{d^2U}{dxdy} + y'^2\frac{d^2U}{dy^2},$$

then it has been proved (Art. 79) that when both x', y'; x, y; are linearly transformed, this will become

$$X'^2\frac{d^2U}{dX^2} + 2X'Y'\frac{d^2U}{dXdY} + Y'^2\frac{d^2U}{dY^2}.$$

Suppose that when $x'y'$ *only* are transformed, it becomes $aX'^2 + 2bX'Y' + cY'^2$, then a, b, c must be functions of x and y, such that when these variables are linearly transformed, a, b, c will become respectively $\dfrac{d^2U}{dX^2}$, $\dfrac{d^2U}{dXdY}$ $\dfrac{d^2U}{dY^2}$. Now it has been proved (Art. 72) that if the same emanant, by transformation of $x'y'$ only, becomes $aX'^2 + 2bX'Y' + cY'^2$ we shall have $ac - b^2 = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 \left\{\dfrac{d^2U}{dx^2}\dfrac{d^2U}{dy^2} - \left(\dfrac{d^2U}{dxdy}\right)^2\right\}$; but now transforming x, y, in a, b, c, we have

$$\frac{d^2U}{dX^2}\frac{d^2U}{dY^2} - \left(\frac{d^2U}{dXdY}\right)^2 = \left\{\frac{d^2U}{dx^2}\frac{d^2U}{dy^2} - \left(\frac{d^2U}{dxdy}\right)^2\right\}(\alpha_1\beta_2 - \alpha_2\beta_1)^2$$

and, therefore, the function

$$\frac{d^2U}{dx^2}\frac{d^2U}{dy^2}-\left(\frac{d^2U}{dxdy}\right)^2$$

is a covariant.

And, in general, if $\left(x'\dfrac{d}{dx}+y'\dfrac{d}{dy}+\&c.\right)^n U$ by transformation of $x'y'$ only becomes $aX'^n + nbX'^{n-1}Y' + \&c.$; then an invariant of this emanant is, by definition, a function of its coefficients which differs only by a power of the modulus from a similar function of a, b, c, &c. But (Art. 79) a, b, c, &c., when x, y, &c. are transformed, become $\dfrac{d^nU}{dX}, \dfrac{d^nU}{dX^{n-1}dY}$, &c., and, therefore, any invariant of the emanant is a function of the differentials of the given quantic, which becomes by transformation a similar function of the differentials of the transformed quantic; that is to say, it is a covariant.

81. It was proved (Art. 77) that every binary quantic of even degree had an invariant of the second order in the coefficients. To every one of these corresponds a covariant of quantics in general; thus we have a series of covariants:—

$$\frac{d^2U}{dx^2}\frac{d^2U}{dy^2}-\left(\frac{d^2U}{dxdy}\right)^2;$$

$$\frac{d^4U}{dx^4}\frac{d^4U}{dy^4}-4\frac{d^4U}{dx^3dy}\frac{d^4U}{dxdy^3}+3\left(\frac{d^4U}{dx^2dy^2}\right)^2;$$

$$\frac{d^6U}{dx^6}\frac{d^6U}{dy^6}-6\frac{d^6U}{dx^5dy}\frac{d^6U}{dxdy^5}+15\frac{d^6U}{dx^4dy^2}\frac{d^6U}{dx^2dy^4}-10\left(\frac{d^6U}{dx^3dy^3}\right)^2 \&c.$$

Of course the second of these is inapplicable to any quantic under the fourth degree; it yields an invariant for that degree, and a covariant for higher degrees: the third of these is inapplicable to any quantic under the sixth degree, and so on.

The first of this series is called the *Hessian** of the quantic. It may also be defined as the Jacobian (see p. 37) of the n

* In general for a quantic in any number of variables the discriminant of the second emanant is called the Hessian. The Hessian of a binary quantic is evidently of the degree $2(n-2)$ in the variables, and of the second degree in the coefficients.

differentials of a quantic. It will be proved further on that the Jacobian of any system of equations is a covariant of that system.

82. Before proceeding further, it may be well to give an example of the use which can be made of the theory of covariants. Let us take the problem of the solution of the cubic equation $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, which could evidently be effected if by linear transformations we could bring it to the form

$$AX^3 + DY^3 = 0,$$

in which shape its resolution into factors would be immediately performed. Now, if by any linear transformation we have

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = AX^3 + 3BX^2Y + 3CXY^2 + DY^3;$$

then, by the definition of a covariant, the Hessian

$$(ax + by)(cx + dy) - (bx + cy)^2; \text{ or}$$

$$(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2$$
$$= (AC - B^2)X^2 + (AD - BC)XY + (BD - C^2)Y^2.$$

Now, if by the transformation the new B and C vanish, the Hessian reduces itself to $ADXY$. It follows, then, that we are to take for X and Y the two factors into which the Hessian can be decomposed; and having thus found X and Y, if we equate the given cubic to $AX^3 + DY^3$, we at once determine A and D by comparison of any two coefficients.

Ex. To reduce to the form $AX^3 + DY^3$, $4x^3 + 9x^2 + 18x + 17$.

The Hessian $(4x + 3)(6x + 17) - (3x + 6)^2 = 15x^2 + 50x + 15$, whose factors are $x + 3$, $3x + 1$; and if we equate the given cubic to $A(x + 3)^3 + D(3x + 1)^3$, we have from the first and last coefficients

$$A + 27D = 4,\ 27A + D = 17,\ 728D = 91,\ 728A = 455;\ A : D :: 5 : 1,$$

and, therefore, the given cubic is equivalent to $5(x + 3)^3 + (3x + 1)^3$.

83. *Every binary quantic of odd degree has a covariant of the second degree in the variables, and in the coefficients.* For the emanant $\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^{n-1}$ is then of even degree in x', y', and its coefficients are of the first degree in x and y. Its invariant then, as in Art. 81, gives us the covariant required. Thus, for a

quintic, the second of the series in Art. 81 will be a quadric; for a septic, the third in that series, and so on.

84. *Every invariant of a covariant is an invariant of the original quantic.*

This follows directly from the definitions: for when the variables are transformed the expression for the invariant in terms of the coefficients of the covariant is unchanged, and the expression of these in terms of the coefficients of the original quantic is unaltered; therefore the expression of the invariant in terms of these latter coefficients is unaltered.

85. *Every binary quantic of odd degree has at least one invariant of the fourth order.*

For it has (Art. 83) a quadratic covariant whose coefficients are of the second degree, and the discriminant of that quadric will be of the fourth order. Thus, for a cubic whose Hessian is $(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2$; the discriminant of that Hessian $(ac - b^2)(bd - c^2) - 4(ad - bc)^2$ is an invariant of the cubic. It is, in fact, its discriminant. It is well known that every cubic cannot, by real transformations, be brought to the form $Ax^3 + Dy^3$; for since this last form has one real and two imaginary factors, a cubic which has three real factors cannot be reduced to that form. Now these differences are indicated by the sign of the discriminant of the Hessian above written. When that discriminant is positive, the Hessian has two real factors, and therefore, the cubic has one real and two imaginary factors. When that discriminant is negative, the Hessian has two imaginary factors, and the cubic three real. When it vanishes, both Hessian and cubic have two equal factors. For if the cubic is of the form $X^2 Y$, its Hessian can easily be seen to be X^2, which has two equal factors also.*

* Generally, if a binary quantic have a square factor, that will be also a square factor in the Hessian, as can easily be proved by forming the Hessian of $x^2\phi$.

This leads us easily to the condition that a quartic should have two square factors. For these must be square factors in the Hessian too, and since the Hessian of a quartic is itself a quartic, the Hessian and quantic can in this case only differ by a numerical factor. By equating, then, coefficients in the quartic and its Hessian, we get the conditions required.

86. From the covariants formed by the rules thus far explained, we can again form new invariants by the method of Art. 76, which, again, by Art. 80, give rise to new covariants. Thus, take the quartic $(a, b, c, d, e \between x, y)^4$, and we have learned that $\left(a, b, c, d, e \between \frac{d}{dy}, -\frac{d}{dx}\right)^4$ is an invariantive symbol of operation. If, then, we operate with this on the Hessian

$$(ax^2 + 2bxy + cy^2)(cx^2 + 2dxy + ey^2) - (bx^2 + 2cxy + dy^2)^2;$$

or

$$(ac - b^2,\ 2ad - 2bc,\ ae + 2bd - 3c^2,\ 2be - 2cd,\ ce - d^2 \between x, y)^4,$$

the result will be $ace + 2bcd - ad^2 - eb^2 - c^3$, which is an invariant of the quartic; and, consequently,

$$\frac{d^4U}{dx^4}\frac{d^4U}{dy^4}\frac{d^4U}{dx^2dy^2} + 2\frac{d^4U}{dx^3dy}\frac{d^4U}{dxdy^3}\frac{d^4U}{dx^2dy^2} - \&c.$$

is a covariant of any quantic above the fourth degree.

87. Before concluding this Lesson, we have one more general principle to lay down, the consequences of which are of great importance. First, let x', y' be any variables cogredient with x and y (see Art. 78), then we say that $xy' - yx'$ is a covariant to any quantic whatever. For when both x, y; x', y'; are linearly transformed, then, by the theorem for multiplication of determinants, $xy' - yx' = (\alpha_1\beta_2 - \alpha_2\beta_1)(XY' - YX')$.

88. Let I be any invariant of a quantic $(a_0, a_1, a_2, \ldots \between x, y)^n$, then

$$y^n\frac{dI}{da_0} - y^{n-1}x\frac{dI}{da_1} + y^{n-2}x^2\frac{dI}{da_2} - \&c.$$

is called an *evectant* of the invariant; or, more generally,

$$\left(y^n\frac{d}{da_0} - y^{n-1}x\frac{d}{da_1} + \&c.\right)^p I$$

is an evectant.

We shall now prove that *every evectant is a covariant to the quantic.* Suppose that by transformation $(a_0, a_1 \ldots \between x, y)^n$ becomes $(A_0, A_1 \ldots \between X, Y)^n$, it follows from the last Article that $(a_0, a_1 \ldots \between x, y)^n + \lambda(xy' - yx')^n$ will become

$$(A_0, A_1 \ldots \between X, Y)^n + \lambda(XY' - YX')^n.$$

But the first, written in full, is

$$(a_0 + \lambda y'^n)x^n + nx^{n-1}y(a_1 - \lambda y'^{n-1}x') + \frac{n(n-1)}{1 \cdot 2}x^{n-2}y^2(a_2 + \lambda y'^{n-2}x'^2) - \&c.$$

If now we form any invariant of the last quantic, it will, when transformed, become a similar function of the new coefficients. But the invariant of this quantic can be formed from the corresponding invariant of the original, by writing in it $a_0 + \lambda y'^n$ for a_0, $a_1 - \lambda y'^{n-1}x'$ for a_1 &c., when it becomes

$$I + \lambda\left(y'^n \frac{d}{da_0} - y'^{n-1}x' \frac{d}{da_1} + \&c.\right) I + \&c.,$$

which is equal to the corresponding function of A_0, A_1, &c., and by equating coefficients of like powers of λ, we see that every evectant of I is changed by linear transformation into a function of similar form; that is to say, it is a covariant.

Ex. A cubic $(abcd \between xy)^3$ has an invariant (Art. 85),

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd;$$

to find its evectant. *Ans.*

$$(a^2d + 2b^3 - 3abc,\ 3b^2c + 3abd - 6ac^2,\ 6b^2d - 3bc^2 - 3acd,\ 3bcd - 2c^3 - ad^2 \between x, y)^3.$$

89. *If a binary quantic have a square factor $(xy' - yx')^2$, the first evectant of the discriminant reduces itself to $(xy' - yx')^n$.*

By taking $xy' - yx' = Y$, we can transform the quantic into a form in which the new A_0, A_1 will both vanish. Now it has been proved (Art. 68) that the discriminant is of the form $a_0\phi + a_1^2\psi$. The differentials then of this discriminant with respect to any other coefficient a_2 will be of the form $a_0\phi' + a_1^2\psi'$, and will, therefore, vanish when a_0 and a_1 both vanish. Also, the differential with respect to a_1 is of the form $a_0\phi' + a_1^2\psi' + 2a_1\psi$, which will also vanish when a_0 and a_1 vanish. The only differential, then, that does not vanish is that with respect to a_0, and the evectant reduces to the single term $Y^n \dfrac{dI}{dA_0}$.

Thus, then, for example, if the discriminant of a cubic vanish, its evectant written in the last example will be a perfect cube, and will give the singular root $xy' - yx'$. It can at once be seen

that the root found by this method is identical with that found by the method of Art. 69.

If more than one pair of roots be equal, $\frac{dI}{da_0}$ also vanishes, and it is proved in the same way that the second evectant of the discriminant

$$\left(y^n \frac{d}{da_0} - y^{n-1}x \frac{d}{da_1} + y^{n-1}x^2 \frac{d}{da_2} - \&c.\right)^2 I$$

is then a perfect n^{th} power, and equal $(xy' - yx')^n(xy'' - yx'')^n$; so that the two equal roots are found by extracting the n^{th} root of this evectant, and then solving a quadratic equation.

LESSON IX.

CONTRAVARIANTS.

90. THERE are some points in which the general theory is disguised when applied to binary quantics. We shall in this Lesson extend to quantics in general the results obtained in the preceding Lesson for binary quantics, but dwelling at length only on those points where the theory is distinct. It is sufficient to consider the case of three variables, as the theory is the same for all higher quantics. We commence by examining how the results of Art. 76 are to be extended; and we investigate, when variables are transformed by the linear substitutions

$$x = \alpha_1 X + \beta_1 Y + \gamma_1 Z,\ y = \alpha_2 X + \beta_2 Y + \gamma_2 Z,\ z = \alpha_3 X + \beta_3 Y + \gamma_3 Z,$$

by what rule the differentials of the quantic are transformed.

From the preceding equations we have (Art. 24)

$$\Delta X = A_1 x + A_2 y + A_3 z,\ \Delta Y = B_1 x + B_2 y + B_3 z,\ \Delta Z = C_1 x + C_2 y + C_3 z:$$

where Δ is the determinant $(\alpha_1\beta_2\gamma_3)$, and A_1, B_1, &c., the minors corresponding to α_1, β_1, &c. Again, we have

$$\frac{d}{dx} = \frac{d}{dX}\frac{dX}{dx} + \frac{d}{dY}\frac{dY}{dx} + \frac{d}{dZ}\frac{dZ}{dz} = \frac{1}{\Delta}\left\{A_1 \frac{d}{dX} + B_1 \frac{d}{dY} + C_1 \frac{d}{dZ}\right\}.$$

In like manner

$$\frac{d}{dy} = \frac{1}{\Delta}\left\{A_2\frac{d}{dX} + B_2\frac{d}{dY} + C_2\frac{d}{dZ}\right\};\ \frac{d}{dz} = \frac{1}{\Delta}\left\{A_3\frac{d}{dX} + B_3\frac{d}{dY} + C_3\frac{d}{dZ}\right\}$$

which are the formulæ required.

91. We have already said (Art. 78) that a set of variables $x'y'z'$ are said to be *cogredient* with xyz, if it is understood that whenever xyz are linearly transformed, $x'y'z'$ are to be transformed by *the same* substitutions. Now, a set of variables $\xi\eta\zeta$ are said to be *contragredient* to xyz if it is understood that whenever xyz are linearly transformed, $\xi\eta\zeta$ are also to be linearly transformed, but by *reciprocal* substitutions (see Art. 25), that is to say, that when we substitute for x, $\alpha_1 x + \beta_1 y + \gamma_1 z$, we are to substitute for ξ, $A_1\xi + B_1\eta + C_1\zeta$ &c. where A_1, B_1, C_1 are the constituents of the reciprocal determinant. The results, then, of the last Article may be briefly expressed by saying that $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, are contragredient to x, y, z. It is, of course, equally true for binary quantics that $\frac{d}{dx}, \frac{d}{dy}$ are contragredient to x, y. If, however, the direct transformation is $\alpha_1 X + \beta_1 Y$ for x, and $\alpha_2 X + \beta_2 Y$ for y; this may be expressed by saying that y is transformed into $\beta_2 Y - \alpha_2(-X)$ and $-x$ into $-\beta_1 Y + \alpha_1(-X)$. But this last is (see Art. 76) the contragredient or reciprocal substitution. Hence y and $-x$ are contragredient to x and y, and instead of saying that $\frac{d}{dx}$ and $\frac{d}{dy}$ are contragredient to x and y, we found it more simple to say that they are cogredient to y and $-x$.

We shall, in what follows, ordinarily use Greek letters ξ, η, ζ, to denote variables contragredient to x, y, z.

92. *The function $x\xi + y\eta + z\zeta$, where $\xi\eta\zeta$ are contragredient to xyz, is unaltered by linear transformation.* For when both are transformed according to the rules explained, the new coefficient of $x\xi$ will be $A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 = \Delta$ (Art. 22), which will also be the new coefficient of $y\eta$ and $z\zeta$; while the coefficient of every other term vanishes,—that of $x\eta$, for instance, being

$B_1a_1 + B_2a_2 + B_3a_3 = 0$ (Art. 23). Hence $x\xi + y\eta + z\zeta$ is transformed into $\Delta(x\xi + y\eta + z\zeta)$. Art. 87 is a particular case of this, ξ and η being cogredient with y' and $-x'$.

93. We have defined a covariant as a derivative which retains its relation to the primitive when the variables in both are transformed by *the same* linear transformations. We have now to define a *contravariant* which is a derivative function of $\xi\eta\zeta$, and which retains its relation to its primitive when the variables in both are transformed by *reciprocal* substitutions. Thus, take the ternary quadric (or the equation of the conic)

$$ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy,$$

then the reciprocal conic is represented (see "Conics," p. 267) by

$$(bc - d^2)\xi^2 + (ca - e^2)\eta^2 + (ab - f^2)\zeta^2 + 2(ef - ad)\eta\zeta$$
$$+ 2(fd - be)\zeta\xi + 2(de - cf)\xi\eta.$$

Now, this is *not* a covariant, for if the conic be referred to any new axes, the reciprocal with regard to the new origin is no longer the same curve as before. But it *is* a contravariant, for it represents (see "Conics," p. 251) the condition that the line $x\xi + y\eta + z\zeta$ should touch the conic, and it has been just proved that when we transform ξ, η, ζ, by reciprocal substitutions, $x\xi + y\eta + z\zeta$ remains unaltered. We must, therefore, get the same result whether we transform ξ, η, ζ, in the quantic $(bc - d^2)\xi^2 +$ &c., or whether we form directly the condition that $x\xi + y\eta + z\zeta$ should touch the transformed of $ax^2 +$ &c.

In like manner, the equation of the reciprocal of any curve is always a contravariant (though, in general, not the *only* contravariant) to the equation of the curve.*

The word *concomitant* is used as a general term to include both covariant and contravariant; and a *mixed concomitant* is a derivative function containing both x, y, z; ξ, η, ζ, which retains

* In general, if $\phi(x, y, z)$ represent a curve, $\psi(\xi, \eta, \zeta)$ denotes a relation between the coefficients of the equation of a right line $(x\xi + y\eta + z\zeta)$ subjected to certain conditions. $\psi(\xi, \eta, \zeta)$ may, therefore, be taken to represent the envelope of the line in question; or, in other words, it is the equation of that envelope in line coordinates. See "Higher Plane Curves," chap. i.

its relation to the primitive, when the one set of variables is transformed by direct, and the other by reciprocal, substitutions.

94. It follows at once from Arts. 90, 91 (see also Art. 76), that if in any contravariant we write $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, for ξ, η, ζ, we shall have an invariantive symbol of operation. Thus, assuming for the present that the function given (Art. 93) $(bc-d^2)\xi^2 + \&c.$ *is* a contravariant to $ax^2 + \&c.$, we may substitute in the former $\frac{d}{dx}$ for ξ, &c., and operating with it then on the latter, we get $abc + 2def - ad^2 - be^2 - cf^2$, which we should thence infer must be an invariant of the given quadric. In like manner, we may substitute $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, for x, y, z.

95. Again, for the same reason, the result of substituting $\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}$, for ξ, η, ζ, is a covariant. Thus, in a binary quantic, the result of substituting $\frac{dU}{dx}, \frac{dU}{dy}$, for y and $-x$ is a covariant. For example, if $U = ax^2 + 2bxy + cy^2$, then

$$a\left(\frac{dU}{dy}\right)^2 - 2b\left(\frac{dU}{dx}\right)\left(\frac{dU}{dy}\right) + c\left(\frac{dU}{dx}\right)^2$$

is a covariant. It is, in fact, proportional to $(ac - b^2)U$.

In like manner, the result of substituting $\frac{dU}{dx}$ for ξ &c. in $(bc - d^2)\xi^2 + \&c.$ is a covariant. It will be found to be

$$(abc + 2def - ad^2 - bc^2 - cf^2)U.$$

It follows from the principles laid down that the Jacobian is a covariant. For when x, y, z are linearly transformed, $\frac{dU}{dx}, \frac{dV}{dx}$, &c., are transformed by the reciprocal substitution; and by the theorem of multiplication of determinants, the determinant formed with the transformed $\frac{dU}{dx}$ &c. is equal to the original Jacobian multiplied by the determinant formed with the reci-

procal constituents A_1, B_1, &c., which (Art. 25) is a power of the modulus of transformation.

96. We may generalize Arts. 88, 89, as follows. If I be any invariant of a quantic

$$a_0x^n + na_1x^{n-1}y + nb_1x^{n-1}z + \frac{n(n-1)}{1 \cdot 2} a_2x^{n-2}y^2 + \&c.,$$

then

$$\left(\xi^n \frac{d}{da_0} + \xi^{n-1}\eta \frac{d}{da_1} + \xi^{n-1}\zeta \frac{d}{db_1} + \xi^{n-2}\eta^2 \frac{d}{da_2} + \&c.\right)^p I$$

is an *evectant* of that invariant. It is to be observed, that the given quantic is written *with*, and the evectant *without*, binomial coefficients. Every evectant is proved to be a contravariant, as at Art. 88. Thus, if a_0x^n + &c. becomes by transformation A_0X^n + &c., then it has been proved (Art. 92) that $x\xi + y\eta + z\zeta$ is a universal concomitant: therefore

$$a_0x^n + \&c. + \lambda(x\xi + y\eta + z\zeta)^n = A_0X^n + \lambda(X\Xi + Y\mathrm{H} + Z\mathrm{Z})^n,$$

and any invariant of one side of the equation will be transformed into a similar function of the quantities on the other. But any invariant of $a_0x^n + \lambda(x\xi + \&c.)^n$ is got from the corresponding invariant of the original quantic by writing $a_0 + \lambda\xi^n$ for a_0, $a_1 + \lambda\xi^{n-1}\eta$ for a_1, $b_1 + \lambda\xi^{n-1}\zeta$ for b_1 &c., and any invariant I becomes thus $I + \lambda \left(\xi^n \frac{d}{da_0} + \&c.\right) I + \&c.$

The coefficients, then, of each power of λ will be transformed into similar functions of the new coefficients, and since it is understood all along that ξ, η, ζ are transformed by the reciprocal substitutions, therefore, the coefficient of λ is a *contravariant*. Thus, if we know that the discriminant of the quadric (Art. 93) is $abc + 2def + ad^2 - be^2 - cf^2$, we can infer that its evectant

$$\xi^2 \frac{dI}{da} + \eta^2 \frac{dI}{db} + \xi\eta \frac{dI}{df} + \&c.$$

is a contravariant. This evectant is $(bc - d^2)\xi^2 + (ca - e^2)\eta^2$ + &c., the same as the equation of the reciprocal already referred to.*

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* There is little doubt that the reciprocal of an algebraic curve can always be represented in the form $\left(\xi^n \frac{d}{da} + \xi^{n-1}\eta \frac{d}{db} + \&c.\right)^{n-1} I$, where I is a certain invariant of the order $3(n-1)$.

97. When the discriminant of a ternary quantic vanishes, it has a set of singular roots, $x'y'z'$ [geometrically, the co-ordinates of the double point on the curve represented by the quantic]; and in this case the first evectant of the discriminant will be a perfect n^{th} power of $x'\xi + y'\eta + z'\zeta$. For if the discriminant vanishes, the quantic can be so transformed that the new coefficients of z^n, $z^{n-1}x$, $z^{n-1}y$, viz., a_0, a_1, b_1 shall vanish [in geometrical language, so that the origin shall be the double point]. But it was proved (Art. 71) that the form of the discriminant is

$$a_0\rho + a_1^2\phi + a_1b_1\psi + b_1^2\chi.$$

It is proved then, as at Art. 89, that when a_0, a_1, b_1 all vanish, every differential coefficient of the discriminant vanishes except $\frac{dI}{da_0}$. The evectant then reduces itself to $\frac{dI}{da_0}$ multiplied by the perfect n^{th} power ξ^n, which is what $(x'\xi + y'\eta + z'\zeta)^n$ becomes when x' and $y' = 0$, and $z' = 1$.

Thus, then, if the discriminant of a quadric vanish; that is, if

$$abc + 2def - ad^2 - be^2 - cf^2 = 0;$$

the quadric represents two right lines, and $x'y'z'$, the co-ordinates of the point of intersection of those lines, are given by the identity

$$(x'\xi + y'\eta + z'\zeta)^2 = (bc - d^2)\xi^2 + (ca - e^2)\eta^2 + \&c.$$

If the curve represented by a ternary quantic have two double points, all the first differentials of the discriminant vanish, and its second evectant becomes a perfect n^{th} power of

$$(x'\xi + y'\eta + z'\zeta)(x''\xi + y''\eta + z''\zeta)$$

where $x'y'z'$, $x''y''z''$ are the two double points.

LESSON X.

DIFFERENTIAL EQUATION OF INVARIANTS.

98. *If n be the order of any binary quantic, θ the order in the coefficients of any of its invariants, then the* weight (*see Art.* 63) *of every term in the invariant is constant and* $= \frac{1}{2}n\theta$.

If we alter x into ρx, leaving y unchanged, then, since this is a linear transformation, the invariant must, by definition, remain unaltered, except that it may be multiplied by a power of ρ which is in this case the modulus of transformation. It is proved then, precisely as in Art. 34, that the *weight*, or sum of the suffixes in every term, is constant.

Again, the invariant must remain unaltered if we change x into y, and y into x, a linear transformation the modulus of which is -1. The effect of this transformation is the same as if for every coefficient a_α we substitute $a_{n-\alpha}$. Hence the sum of a number of suffixes

$$\alpha + \beta + \gamma + \&c. = (n - \alpha) + (n - \beta) + (n - \gamma) + \&c.,$$

whence $2(\alpha + \beta + \gamma + \&c.) = n\theta$. Q. E. D.*

The principles established in this article enable us to write down immediately the literal part of any invariant whose order is given. Thus, if we are required to form for a cubic an invariant of the fourth order in the coefficients, its weight must $= 6$, and the terms must be

$$a_3a_3a_0a_0,\ a_3a_2a_1a_0,\ a_3a_1a_1a_1,\ a_2a_2a_2a_0,\ a_2a_2a_1a_1.$$

We shall presently show how the coefficients of these terms can be found.

The reader will observe that there are as many terms in this

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* It follows immediately, that n and θ cannot both be odd, since their product is an even number: or, *a quantic of odd degree cannot have an invariant of an odd order.*

invariant as the ways in which the number 6 can be expressed as the sum of four numbers from 0 to 3 inclusive; and generally, that there may be as many terms in any invariant as the ways in which its weight $\frac{1}{2}n\theta$ can be expressed as the sum of θ numbers from 0 to n inclusive.

99. Similar reasoning applies to covariants. A covariant like the original quantic must remain unaltered, when we change x into ρx and at the same time every coefficient a_a into $\rho^a a_a$. If, then, the coefficient of any power of x, x^μ in the covariant be $a_a b_\beta c_\gamma$ &c., it is obvious, as before, that $\mu + a + \beta + \gamma +$ &c. must be constant for every term; and we may call this number the *weight* of the covariant.

Again, in order that the covariant may not change when we alter x into y, and y into x, we must have

$$\mu + a + \beta + \gamma + \&c. = (n' - \mu) + (n - a) + (n - \beta) + \&c.,$$

where n' is the degree of the covariant in x and y: whence, if θ be the order of the covariant in the coefficients, we have immediately its weight $= \frac{1}{2}(n\theta + n')$. Thus, if it were required to form a quadratic covariant to a cubic, of the second degree in the coefficients, the weight is $= 4$: we have, then, for the terms multiplying x^2, $a + \beta = 2$, and the terms must be a_2a_0 and a_1a_1. In like manner, the terms multiplying xy must be a_3a_0 and a_2a_1, and those multiplying y^2 must be a_3a_1 and a_2a_2. We proceed now to establish the principle by which we can determine the coefficients of the terms, the literal part of which we have shown how to find in this and in the last Article.

100. Every invariant satisfies certain differential equations which can be found from very simple considerations. Take the binary quantic $(a_0, a_1, a_2 \ldots \between x, y)^n$. Then any invariant is, by definition, a function of its coefficients which will not alter when we change x into $x + \lambda y$, leaving y unaltered.* But the effect of this substitution is to make the quantic become

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* In this case the modulus of transformation is $\begin{vmatrix} 1, \lambda \\ 0, 1 \end{vmatrix} = 1.$

$$a_0x^n + n(a_1 + \lambda a_0)x^{n-1}y + \frac{n(n-1)}{1 \, . \, 2}(a_2 + 2\lambda a_1 + \lambda^2 a_0)x^{n-2}y^2 + \&c.$$

Any function, then, of the coefficients of the transformed quantic is got from the corresponding function of the original by writing $a_1 + \lambda a_0$ for a_1, $a_2 + 2\lambda a_1 + \lambda^2 a_0$ for a_2, &c. But if the function is an invariant, the coefficients of the several powers of λ must vanish; and, confining our attention for the present to the coefficient of λ, we see that an invariant must satisfy the condition

$$a_0 \frac{dI}{da_1} + 2a_1 \frac{dI}{da_2} + 3a_2 \frac{dI}{da_3} \dots\dots + na_{n-1} \frac{dI}{da_n} = 0.$$

Thus, for example, for the invariant of a quadric $a_0a_2 - a_1^2$, we have

$$\frac{dI}{da_1} = -2a_1, \; \frac{dI}{da_2} = a_0; \quad a_0 \frac{dI}{da_1} + 2a_1 \frac{dI}{da_2} = 0.$$

It may be proved in like manner, by keeping x unaltered, and changing y into $y + \mu x$, that an invariant must also satisfy the equation

$$na_1 \frac{dI}{da_0} + (n-1)a_2 \frac{dI}{da_1} + (n-2)a_3 \frac{dI}{da_2} + \&c. = 0.$$

If the quantic had been written without binomial coefficients $(a_0, a_1, a_2 \dots \between x, y)^n$ it is proved, in like manner, that its invariants must satisfy the equations

$$na_0 \frac{dI}{da_1} + (n-1)a_1 \frac{dI}{da_2} + (n-2)a_2 \frac{dI}{da_3} + \&c. = 0; \; a_1 \frac{dI}{da_0} + 2a_2 \frac{dI}{da_1} + \&c. = 0.$$

101. The conditions just found for an invariant are not only necessary, but sufficient; that is to say, when the coefficient of λ vanishes, the coefficients of the other powers of λ will also vanish. Thus, the coefficient of λ^2 is, without difficulty, found to be

$$a_0 \frac{dI}{da_2} + 3a_1 \frac{dI}{da_3} + 6a_2 \frac{dI}{da_4} + \&c. + \frac{1}{1 \, . \, 2}\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.\right)^2 I;$$

where in the latter symbol the a_0, a_1, &c. which appear explicitly

are not to be differentiated. But it will be seen that this is precisely

$$\frac{1}{1\,.\,2}\left(a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+3a_2\frac{d}{da_3}+\&c.\right)\left(a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+3a_2\frac{d}{da_3}+\&c.\right)I.$$

For when we operate with the symbol on itself, the result will be the sum of the terms got by differentiating the a_1, a_2, &c., which appear explicitly, together with the result, on the supposition that these a_1, a_2, &c., are constant. Thus, then, the coefficient of λ^2 vanishes, since $a_0\dfrac{dI}{da_1}+2a_1\dfrac{dI}{da_2}+\&c.$ is supposed to vanish identically. So, in like manner, for the coefficients of the other powers of λ.

It follows, then, that when the two equations given in the last Article are satisfied, the quantity I will remain unaltered, however the variables are linearly transformed; and, therefore, that these equations are sufficient to determine an invariant. Further, as one of these equations is derived from the other by changing each coefficient a_a into a_{n-a}, it is sufficient to use one of the equations, provided we take care that the function we form is symmetrical with regard to x and y; that is to say, which does not change when we change each a_a into a_{n-a}. This condition will be sufficiently fulfilled if we take care that the weight of the invariant is that determined in Art. 98.

102. The following examples will sufficiently illustrate how the condition just established enables us to form invariants of any given order. We abstain from entering into detail as to methods by which the work may in practice be shortened, since the reader, after working a few examples of this kind, would soon discover such expedients for himself.

Ex. 1.—Let us proceed with the example (Art. 98), and form the invariant of a cubic, the literal part of which we there saw was of the form

$$Aa_3a_3a_0a_0+Ba_3a_2a_1a_0+Ca_3a_1a_1a_1+Da_2a_2a_2a_0+Ea_2a_2a_1a_1.$$

One of the coefficients, A, may be taken $=1$, and it remains to determine B, C, D, E. Operate on the above with the symbol

$$a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+3a_2\frac{d}{da_3}$$

when we get

$$(B+6A)\,a_3a_2a_0a_0+(3C+2B)\,a_3a_1a_1a_0+(2E+6D+3B)\,a_2a_2a_1a_0+(4E+3C)\,a_1a_1a_1a_1=0.$$

Equating separately to 0 the coefficient of each term, and taking $A=1$, we find $B=-6$, $C=4$, $D=4$, $E=-3$.

Ex. 2.—To form an invariant of the third order for a quartic. Here the weight is $=6$, and the invariant is

$$Aa_4a_2a_0+Ba_4a_1a_1+Ca_3a_3a_0+Da_3a_2a_1+Ea_2a_2a_2.$$

Operating with $a_0\dfrac{d}{da_1}+2a_1\dfrac{d}{da_2}+3a_2\dfrac{d}{da_3}+4a_3\dfrac{d}{da_4}$, we find

$$(2B+2A)\,a_4a_1a_0+(D+6C+4A)\,a_3a_2a_0+(2D+4B)\,a_3a_1a_1+(6E+3D)\,a_2a_2a_1=0.$$

Whence we have

$$A=1,\ B=-1,\ D=2,\ E=-1,\ C=-1.$$

Ex. 3.—To form the discriminant of $(a_0, a_1, a_2 \ldots \between x, y)^n$, which we suppose arranged according to powers of a_0. We know already (Art. 68) that the absolute term is a_1^2D, where D is the discriminant of $(a_1, a_2 \ldots \between x, y)^{n-1}$. The discriminant then is

$$a_1^2D+a_0\phi+a_0^2\psi+\&c.$$

Operating on this with $a_1\dfrac{d}{da_0}+2a_2\dfrac{d}{da_1}+3a_3\dfrac{d}{da_2}+\&c.$, we may equate separately to zero the coefficient of each power of a_0. Thus, then, the part independent of a_0 is

$$a_1\phi+4a_1a_2D+a_1^2\left(2a_2\frac{d}{da_1}+3a_3\frac{d}{da_2}+\&c.\right)D;$$

or, remembering that $\left(a_2\dfrac{d}{da_1}+2a_3\dfrac{d}{da_2}+\&c.\right)D=0$, we have

$$\phi=-4a_2D+a_1\left(a_3\frac{d}{da_2}+2a_4\frac{d}{da_3}+\&c.\right)D,$$

and the discriminant is

$$(a_1^2-4a_0a_2)D+a_1a_0\left(a_3\frac{d}{da_2}+2a_4\frac{d}{da_3}+\&c.\right)D+a_0^2\psi+\&c.$$

In like manner, from the coefficient of a_1 we can determine ψ, but the result is not simple enough to seem worth writing down.

103. Mr. Cayley has succeeded in deriving from the preceding theory the number of independent invariants of a binary quantic of given degree. We omit the details of his investigation as too difficult, and merely indicate the principles on which it rests.

In seeking to determine an invariant of given order as in the

examples (Art. 102), we have a certain number of unknown coefficients A, B, C, D, &c., to determine, and we do so by the help of a certain number of conditions formed by the help of the differential equation of the invariant. Now evidently if the number of these conditions were greater than the number of unknown coefficients, the formation of the invariant would be impossible; if they were equal, we could form one invariant; if the number of conditions were less, we could form more than one invariant of the given order. Now, we have seen (Art. 98), that the number of terms in the invariant, which is one more than the number of unknown coefficients, is equal to the number of ways in which the weight $\frac{1}{2}n\theta$ can be written as the sum of θ numbers, none being greater than n. And on looking at any of the examples (Art. 102), the reader will see that the number of conditions is equal to the number of terms in a function of the order θ, and whose weight is one less than the given weight; and that it is therefore equal to the number of ways in which $\frac{1}{2}n\theta - 1$ can be expressed as the sum of θ numbers, none exceeding n. The determination, then, of the number of invariants of a given order depends on a calculation of the difference between these two numbers, and evidently would lead us to an investigation of the theory of the partition of numbers, a subject which we do not mean to include in this series of Lessons.

104. We can, in like manner, find certain differential equations which must be satisfied by the terms of a covariant. From the definition of a covariant we must get the same result, whether we change in it x into $x + \lambda y$, or whether we make the same change in the original quantic, and then form its invariant. But this change in the original quantic (Art. 100) is equivalent to changing a_1 into $a_1 + \lambda a_0$, a_2 into $a_2 + 2a_1\lambda + a_0\lambda^2$, &c. Hence in the covariant also the change of x to $x + \lambda y$ must be equivalent to changing a_1 into $a_1 + \lambda a_0$ &c. Let the covariant then be

$$A_0x^p + pA_1x^{p-1}y + \frac{p(p-1)}{1 \cdot 2}A_2x^{p-2}y^2 + \&c.$$

Let us express that these two alterations are equivalent, and confine our attention to the terms multiplying λ, when we get

$$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \&c.\right) A_0 = 0,$$

$$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \&c.\right) A_1 = A_0,$$

$$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \&c.\right) A_2 = 2A_1,$$

$$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \&c.\right) A_3 = 3A_2, \ \&c.,$$

which are the required equations.

Ex. To find the quadratic covariant of a cubic.

The coefficient of x^2 is $A_0 = a_2a_0 + Ba_1a_1$ (see Art. 99). Operating on this with $a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2}$, the first of the above written conditions gives us $(2 + 2B)a_0a_1 = 0$, whence $B = -1$. In like manner the terms multiplying xy are $2A_1 = Ca_3a_0 + Da_2a_1$; and the second of the above written conditions gives us

$$(D + 3C - 2)a_2a_0 + (2D + 2)\, a_1a_1 = 0;$$

whence $D = -1$, $C = 1$. Lastly, the terms multiplying y^2 being $A_2 = Ea_3a_1 + Fa_2a_2$; the third condition gives $(E - 1)\, a_0a_3 + (4F + 3E + 1)\, a_1a_2 = 0$, whence $E = 1$, $F = -1$, and the required covariant is

$$(a_0a_2 - a_1a_1)x^2 + (a_0a_3 - a_1a_2)\, xy + (a_1a_3 - a_2a_2)y^2.$$

105. The results here arrived at may be stated a little differently. The operation $y \frac{d}{dx}$ performed on any quantic is equivalent to a certain operation performed by differentiating with regard to the coefficients. Thus, for the quantic $(a_1, a_2 \ldots \between x, y)^n$ we get the same result whether we operate on it with $y \frac{d}{dx}$ or with $a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \&c.$ This latter operation, then, may be written $\left[y \frac{d}{dx}\right]$: and the property already proved for a covariant may be written that we have for it $y \frac{d}{dx} - \left[y \frac{d}{dx}\right] = 0$. In other words, that we get the same result whether we operate on the covariant with $y \frac{d}{dx}$ or with $a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \&c.$ In

his latest memoirs Mr. Cayley has started with this property as his *definition* of a covariant: a definition which includes invariants also, since for them we have $y\dfrac{dI}{dx} = 0$; and therefore also, $\left[y\dfrac{dI}{dx}\right] = 0$. We have preferred to follow the historical order, and to consider covariants and invariants from the point of view from which they were first studied.

It can be proved in like manner, that quantics in any number of variables satisfy differential equations which can be expressed in the form $y\dfrac{d}{dx} = \left[y\dfrac{d}{dx}\right]$, $z\dfrac{d}{dx} = \left[z\dfrac{d}{dx}\right]$, &c. Thus, for the quantic $(a, b, c, d, e, f\between x, y, z)^2$ we have

$$y\frac{d}{dx} = a\frac{d}{df} + e\frac{d}{dd} + 2f\frac{d}{db},\quad z\frac{d}{dx} = a\frac{d}{de} + f\frac{d}{dd} + 2e\frac{d}{dc};$$

and any invariant must satisfy the two conditions

$$a\frac{dI}{df} + e\frac{dI}{da} + 2f\frac{dI}{db} = 0,\quad a\frac{dI}{de} + f\frac{dI}{dd} + 2e\frac{dI}{dc} = 0,$$

as may easily be proved from the consideration that the invariant is to remain unaltered if we substitute for x, $x + \lambda y$, or $x + \kappa z$.*

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* It ought to have been mentioned, that in calculating by the method illustrated (Art. 102), the invariants of a quantic in any number of variables, a useful verification is afforded by the fact that *the sum of the numerical coefficients must* = 0, it being supposed that the original quantic was written with binomial coefficients. This will be proved if we show that every invariant must vanish identically if we make all the literal coefficients equal, in which case the quantic will become a perfect n^{th} power, and can by transformation be made to assume the form A_0X^n. Now the law of the weight of the terms of an invariant shows easily that no invariant can contain a term such as $a_0{}^p$. In fact, if it did, it should also by symmetry contain a term such as $b_0{}^p$, where b_0 is the coefficient of y^n, and $a_0{}^p$, $b_0{}^p$, having different weights, cannot be terms in the same invariant. It follows, then, that if we make all the literal coefficients = 0, except a_0, the invariant must vanish.

LESSON XI.

THE HYPERDETERMINANT CALCULUS.

106. When Mr. Cayley first discovered invariants, he gave them the name of "hyperdeterminants;" and, now that the latter name has been replaced by one more expressive, I employ the word "hyperdeterminant" with exclusive reference to a fertile method of investigating invariants and covariants, given by Mr. Cayley in his early researches on this subject.

Let x_1, y_1; x_2, y_2 be any two cogredient sets of variables; then it was proved (Art. 87) that $x_1y_2 - y_1x_2$ is an invariant. Also, since $\frac{d}{dx_1}, \frac{d}{dy_1}$; $\frac{d}{dx_2}, \frac{d}{dy_2}$ are cogredient with each other, both being transformed by the reciprocal substitutions (Art. 76), it follows that $\frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}$ is an invariantive symbol of operation; and that if we operate with any power of this symbol on any function of x_1, y_1, x_2, y_2, we shall obtain a covariant of that function. We shall use for $\frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}$, the abbreviation $\overline{12}$.

Suppose now that we are given any two binary quantics U, V, we can at once form covariants of this system of two quantics. For we have only to write the variables in U with the suffix (1), those in V with the suffix (2), and then operate on the product UV with any power of the symbol $\overline{12}$; when the result must be an invariant or covariant. Thus if

$$U = ax_1^2 + 2bx_1y_1 + cy_1^2; \quad V = a'x_2^2 + 2b'x_2y_2 + c'y_2^2,$$

and that we operate on UV with $\overline{12}^2$, which, written at full length, is

$$\frac{d^2}{dx_1^2}\frac{d^2}{dy_2^2} + \frac{d^2}{dx_2^2}\frac{d^2}{dy_1^2} - 2\frac{d^2}{dx_1dy_1}\frac{d^2}{dx_2dy_2},$$

the result is $ac' + ca' - 2bb'$, which is thus proved to be an inva-

riant of the system of equations. In general, it is obvious that the differentials marked with the suffix (1) only apply to U, and those with the suffix (2) only to V; and it is unnecessary to retain the suffixes after differentiation;* so that $\overline{12}^2$ applied to two quantics of any degree gives the covariant

$$\frac{d^2U}{dx^2}\frac{d^2V}{dy^2}+\frac{d^2U}{dy^2}\frac{d^2V}{dx^2}-2\frac{d^2U}{dxdy}\frac{d^2V}{dxdy}.$$

Similarly the symbol $\overline{12}^3$ applied to two cubics gives the invariant

$$(ad'-a'd)-3(bc'-b'c),$$

or to any two quantics gives the covariant

$$\frac{d^3U}{dx^3}\frac{d^3V}{dy^3}-3\frac{d^3U}{dx^2dy}\frac{d^3V}{dxdy^2}+3\frac{d^3U}{dxdy^2}\frac{d^3V}{dx^2dy}-\frac{d^3U}{dy^3}\frac{d^3V}{dx^3};$$

and so in like manner for the other powers of $\overline{12}$.

107. We can by this method obtain also invariants or covariants of a single function U. It is, in fact, only necessary to suppose in the last article the quantics U and V to be identical. Thus, for instance, in the example of the two quadratics given in the last Article, if we make $a=a'$, $b=b'$, $c=c'$, the invariant $\overline{12}^2$ becomes $2(ac-b^2)$. And, in like manner, the expression there given for the covariant $\overline{12}^2$ of a system U, V, by making $U=V$, gives the covariant of a single quantic

$$\frac{d^2U}{dx^2}\frac{d^2U}{dy^2}-\left(\frac{d^2U}{dxdy}\right)^2.$$

In general, whenever we want by this method to form the covariants of a single function, we resort to this artifice:—We

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* If W be any function containing x_1, y_1; x_2, y_2; we shall get the same result whether we linearly transform these variables, and afterwards omit all the suffixes in the transformed equation; or whether we had omitted the suffixes first, and afterwards transformed x and y. This results immediately from the fact that x_1, y_1; x_2, y_2; x, y, are cogredient. It follows then at once that if W written as a function of x_1, y_1; x_2, y_2; be a covariant of U, V; that is to say, if the expression of the coefficients of W in terms of the coefficients of U and V be unaffected by transformation, then W is also a covariant when the suffixes are all omitted.

first form a covariant of a system of distinct quantics, and then suppose the quantics to be made identical. And in what follows, when we use such symbols as $\overline{12}^n$ &c. without adding any subject of operation, we mean to express derivatives of a single function U. We take for the subject operated on, the product of two or more quantics U_1, U_2, &c., where the variables x_1y_1, x_2y_2, &c. are written in each respectively, instead of x and y; and we suppose that after differentiation all the suffixes are omitted, and that the variables, if any remain, are all made equal to x and y.

108. From the omission of the suffixes after differentiation, it follows at once that it cannot make any difference what figures had been originally used, and that $\overline{12}^n$ and $\overline{34}^n$ are expressions for the same thing. In the use of this method we have constantly to employ transformations depending on this obvious principle. Thus, we can show that when n is odd, $\overline{12}^n$ applied to a single function vanishes identically. For, from what has been said $\overline{12}^n = \overline{21}^n$; but $\overline{12}$ and $\overline{21}$ have opposite signs, as appears immediately on writing at full length the symbol for which $\overline{12}$ is an abbreviation. It follows then that $\overline{12}^n$ must vanish when n is odd. Thus, in the expansion of $\overline{12}^3$, given at the end of Art. 106, if we make $U = V$, it will obviously vanish identically. The series $\overline{12}^2$, $\overline{12}^4$, $\overline{12}^6$ &c. gives the series of invariants and covariants which we have already found (Arts. 77, 81). It is easy to see that $\overline{12}^n$ applied to $(a_0, a_1, a_2 \ldots \between x, y)^n$ gives

$$a_0a_n - na_1a_{n-1} + \frac{n \,.\, n-1}{1 \,.\, 2} a_2a_{n-2} - \&c.,$$

where the last coefficient must be divided by two, as is evident from the manner of formation.

109. The results of the preceding Articles are extended without difficulty to any number of functions. We may take any number of quantics U, V, W, &c., and, writing the variables in the first with the suffix (1), those in the second with the suffix (2), in the third with the suffix (3), and so on, we may operate on their product with the product of any number of symbols $\overline{12}^\alpha$, $\overline{23}^\beta$, $\overline{31}^\gamma$, $\overline{14}^\delta$, &c.; where, as before, $\overline{23}$ is an abbreviation

for $\frac{d}{dx_2}\frac{d}{dy_3} - \frac{d}{dx_3}\frac{d}{dy_2}$, &c. After the differentiation we suppress the suffixes, and we thus get a covariant of the given system of quantics, which will be an invariant if it happens that no power of x and y appear after differentiation. Any number of the quantics U, V, W, &c., may be identical; and in the case with which we shall be most frequently concerned, viz., where we wish to form derivatives of a single quantic, the subject operated on is $U_1U_2U_3$ &c., where U_1 and U_2 only differ by having the variables written with different suffixes.

It is evident that in this method the degree of the derivative in the coefficients will be always equal to the number of different figures in the symbol for the derivative. For if all the functions were distinct, the derivative would evidently contain a coefficient from every one of the quantics U, V, W, &c.; and it will be still true, when U, V, W are supposed identical, that the degree in the coefficients is equal to the number of factors in the product $U_1U_2U_3$ &c. which we operate on. Thus, the derivatives considered in the last Article being all of the form $\overline{12}^p$ are all of the second degree in the coefficients.

Again, if it were required to find the degree of the derivative in x and y. Suppose, in the first place, that the quantics were distinct, U being of the degree n, V of the degree n', W of the degree n'', and so on; and suppose that in the operating symbol the figure 1 occurs α times; 2, β times; and so on; then, since U is differentiated α times; V, β times, &c., the result is of the degree $(n-\alpha)+(n'-\beta)+(n''-\gamma)+$ &c. When the quantics are identical, if there are p factors in the product $U_1U_2 \dots U_p$ which we operate on, the degree of the result in x and y will be $np-(\alpha+\beta+\gamma+\&c.)$. While again, if there be r factors such as $\overline{12}$ in the operating symbol, it is obvious that $\alpha+\beta+\gamma+\&c. = 2r$. It is clear that if we wish to obtain an *in*variant, we must have $\alpha=\beta=\gamma=n$.

110. To illustrate the above principles, we make an examination of all possible invariants of the third degree in the coefficients. Since the symbol for these can only contain three figures

its most general form is $\overline{12}^{\alpha}.\overline{23}^{\beta}.\overline{31}^{\gamma}$; while, in order that it should yield an *in*variant, we must have

$$\alpha + \gamma = \alpha + \beta = \beta + \gamma = n,$$

whence $\alpha = \beta = \gamma$. The general form, then, that we have to examine is $(\overline{12}.\overline{23}.\overline{31})^{\alpha}$. Again, if α be odd, this derivative vanishes identically; for, as in Art. 108, by interchanging the figures 1 and 2, we have $(\overline{12}.\overline{23}.\overline{31})^{\alpha} = (\overline{21}.\overline{13}.\overline{32})^{\alpha}$; but these have opposite signs. It follows, then, that all invariants of the third order are included in the formula $(\overline{12}.\overline{23}.\overline{31})^{\alpha}$, where α is even. Thus, $\overline{12}^2.\overline{23}^2.\overline{31}^2$ is an invariant of a quartic, since the differentials rise to the fourth degree; $\overline{12}^4.\overline{23}^4.\overline{31}^4$ is an invariant of an octavic; $\overline{12}^6.\overline{23}^6.\overline{31}^6$ of a quantic of the twelfth degree, and so on; only quantics whose degree is of the form $4m$ having invariants of the third order in the coefficients. If we wish actually to calculate one of these, suppose $\overline{12}^2.\overline{23}^2.\overline{31}^2$, I write, for brevity, ξ_1, η_1, &c., instead of $\frac{d}{dx_1}$, $\frac{d}{dy_1}$, &c. Then we have actually to multiply out

$$(\xi_1\eta_2 - \xi_2\eta_1)^2 (\xi_2\eta_3 - \xi_3\eta_2)^2 (\xi_3\eta_1 - \xi_1\eta_3)^2.$$

In the result we omit all the suffixes, and replace ξ^4 by $\frac{d^4U}{dx^4}$ &c.; or, when we operate on a quartic, by a_0 the coefficient of x^4. There are many ways which a little practice suggests for abridging the work of this expansion, but we do not think it worth while to give up the space necessary to explain them; and we merely give the results of the expansion of the three invariants just referred to. $\overline{12}^2.\overline{23}^2.\overline{31}^2$ yields the invariant of a quartic already obtained (Arts. 86 and 102), viz.:—

$$a_4a_2a_0 + 2a_3a_2a_1 - a_4a_1^2 - a_0a_3^2 - a_2^3.$$

$\overline{12}^4.\overline{23}^4.\overline{31}^4$ gives

$$a_8(a_4a_0 - 4a_3a_1 + 3a_2a_2) + a_7(-4a_5a_0 + 12a_4a_1 - 8a_3a_2) + a_6(3a_6a_0 - 8a_5a_1 - 22a_4a_2 + 24a_3a_3) + a_5(24a_5a_2 - 36a_4a_3) + 15a_4a_4a_4.$$

And $\overline{12}^6.\overline{23}^6.\overline{31}^6$ gives

$a_{12}(a_6a_0 - 6a_5a_1 + 15a_4a_2 - 10a_3a_3) + a_{11}(-6a_7a_0 + 30a_6a_1 - 54a_5a_2 + 30a_4a_3) + a_{10}(15a_8a_0 - 54a_7a_1 + 24a_6a_2 + 150a_5a_3 - 135a_4a_4) + a_9(-10a_9a_0 + 30a_8a_1 + 150a_7a_2 - 430a_6a_3 + 270a_5a_4) + a_8(-135a_8a_2 + 270a_7a_3 + 495a_6a_4 - 540a_5a_5) + a_7(-540a_7a_4 + 720a_6a_5) - 280a_6a_6a_6$.

111. Though the above-mentioned is the only type of invariants of the third order, there is an unlimited number of covariants, the simplest being $\overline{12}^2.\overline{13}$, which, when expanded, is

$$\frac{d^3U}{dx^3}\frac{d^2U}{dy^2}\frac{dU}{dy} - \frac{d^3U}{dx^2dy}\left(2\frac{d^2U}{dxdy}\frac{dU}{dy} + \frac{d^2U}{dy^2}\frac{dU}{dx}\right) + \frac{d^3U}{dxdy^2}\left(\frac{d^2U}{dx^2}\frac{dU}{dy} + 2\frac{d^2U}{dxdy}\frac{dU}{dx}\right) - \frac{d^3U}{dy^3}\frac{d^2U}{dx^2}\frac{dU}{dx}.$$

When this is applied to a cubic, it gives the evectant obtained already, page 62.

The general type of invariants of the fourth order in the coefficients is $(\overline{12}.\overline{34})^\alpha(\overline{13}.\overline{24})^\beta(\overline{14}.\overline{23})^\gamma$. Thus the discriminant of a cubic is expressed in this notation $(\overline{12}.\overline{34})^2(\overline{13}.\overline{24})$; but we cannot afford space to enter into greater details on this subject.

112. The principles just laid down afford an easy proof of a remarkable theorem first demonstrated by M. Hermite, and to which we shall refer as "Hermite's Law of Reciprocity." *The number of invariants of the n^{th} degree in the coefficients possessed by a binary quantic of the p^{th} degree is equal to the number of invariants of the degree p in the coefficients possessed by a quantic of the n^{th} degree.* We have already proved that if any symbol $\overline{12}^a.\overline{23}^b.\overline{34}^c$ &c. denotes an invariant of the order p of a quantic of the degree n; then the number of different figures 1, 2, 3, &c., is p, and each figure occurs n times. But we might calculate by the method of Art. 75 an invariant $\Sigma(\alpha-\beta)^a(\beta-\gamma)^b(\gamma-\delta)^c$ &c., where we replace each symbol $\overline{34}$ by the difference of two roots $(\gamma-\delta)$. This latter is an invariant of a quantic of the p^{th} degree, since there are by hypothesis p roots; and it is of the degree n in the coefficients of the equation; for it can be easily proved that the degree in the coefficients of any symmetrical function of the roots is equal to the highest power in which any root enters into the expression for the function.

Thus, for example, a quadratic has but the single independent invariant $(\alpha - \beta)^2$, though of course every power of this is also an invariant; and the general type of such invariants is $(\alpha - \beta)^{2m}$. Hence, only quantics of even degree have invariants of the second order in the coefficients, and the general symbol for such invariants is $\overline{12}^{2m}$.

So again, cubics have no invariant except the discriminant $(\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2$ and its powers; and the discriminant is of the fourth order in the coefficients. Hence only quantics of the degree $4m$ have cubic invariants whose general type is $\overline{12}^{2m}.\overline{23}^{2m}.\overline{31}^{2m}$. It will be proved that quartics have two independent invariants, one of the second, and one of the third degree, in the coefficients; and, of course, any power of one multiplied by any power of the other is an invariant. Hence, quartics have as many invariants of the p^{th} order as the equation $2x + 3y = p$ admits of integer solutions: this is, therefore, the number of invariants of the fourth order which a quantic of the p^{th} degree can possess.

113. Hermite has proved that his theorem applies also to covariants of any given degree in x and y; that is to say, that an n^{ic} possesses as many such covariants of the p^{th} order in the coefficients as a p^{ic} has of the n^{th} order in the coefficients. For, consider any symbol, $\overline{12}^{\lambda}.\overline{23}^{\mu}.\overline{34}^{\nu}$ &c., where there are p figures, and the figure 1 occurs a times, 2 occurs b times, and so on; then we have proved that the degree of this covariant in x and y is $(n - a) + (n - b) +$ &c. But we may form the symmetric function

$$\Sigma (\alpha - \beta)^{\lambda} (\beta - \gamma)^{\mu} (\gamma - \delta)^{\nu} (x - \alpha)^{n-a} (x - \beta)^{n-b} \text{ \&c.},$$

which has been proved (Art. 75) to be a covariant of the quantic of the p^{th} degree, whose roots are α, β, &c. Every root enters into its expression in the degree n, which is therefore the degree of the covariant in the coefficients, and it obviously contains x and y in the same degree as before, viz. $(n - a) + (n - b) +$ &c. Thus, for example, the only covariants which a quadratic has, are some power of the quantic itself multiplied by some power of its discriminant, the general type of which is

$$(\alpha - \beta)^{2p}(x - \alpha)^q(x - \beta)^q,$$

the order of which in the coefficients is $2p + q$, and in x and y is $2q$. Hence we infer that every quantic of the degree $2p + q$ has a covariant of the second degree in the coefficients, and of the degree $2q$ in x and y, the general symbol for such covariants being 12^{2p}. When $q = 1$, we obtain the theorem given (Art. 83), that every quantic of odd degree has a quadratic covariant.

114. This hyperdeterminant notation affords a complete calculus, by means of which invariants or covariants can be transformed, and the identity of different expressions ascertained. Postponing to a subsequent Lesson the explanation of this, we go on to show how the same notation is to be applied to express derivatives of quantics in three or more variables. If $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, be cogredient sets of variables, then, by the rule for multiplication of determinants, the determinant

$$x_1(y_2z_3 - y_2z_3) + x_2(y_3z_1 - z_3x_1) + x_3(y_1z_2 - y_2z_1)$$

is an invariant, which by transformation, becomes a similar function multiplied by the modulus of transformation. And if in the above we write for x_1, $\dfrac{d}{dx_1}$; for y_2, $\dfrac{d}{dy_2}$; and so on, we obtain an invariantive symbol of operation, which we shall write $\overline{123}$. When, then, we wish to obtain invariants or covariants of any function U, we have only to operate on the product $U_1U_2U_3 \ldots U_p$ with the product of any number of symbols $\overline{123}^\alpha\,\overline{124}^\beta\,\overline{235}^\gamma$ &c., and after differentiation suppress all the suffixes. Thus, for example, let U_1, U_2, U_3 be ternary quadrics, and let the coefficients in U_1 be a, b, c, $2d$, $2e$, $2f$, as at p. 43, then $\overline{123}^2$ expanded is

$$\begin{aligned} a(b'c'' + b''c') + a'(b''c + bc'') + a''(bc' + b'c) + 2d(e'f'' + e''f') \\ + 2d'(e''f + ef'') + 2d''(ef' + e'f) - a(d' + d'')^2 - a'(d'' + d)^2 \\ - a''(d + d')^2 - b(e' + e'')^2 - b'(e'' + e)^2 - b''(e + e')^2 - c(f' + f'')^2 \\ - c'(f'' + f)^2 - c''(f + f')^2; \end{aligned}$$

which, when we suppose the three quantics U_1, U_2, U_3, to be identical, or $a = a' = a''$ &c. reduces to

$$abc + 2def - ad^2 - be^2 - cf^2.$$

If in the above we replace a, the coefficient of x^2, by $\frac{d^2U}{dx^2}$ &c. we get the expansion of $\overline{123}^2$ as applied to any curve. This covariant is called the Hessian of the curve.

It is seen, as at Art. 108, that odd powers of the symbol $\overline{123}$ vanish when it is applied to a single curve. We give as a further example the expansion of $\overline{123}^4$ applied to the quartic,

$$ax^4 + by^4 + cz^4 + 4(a_2x^3y + a_3x^3z + b_3y^3z + b_1y^3x + c_1z^3x + c_2z^3y) \\ + 6(dy^2z^2 + ez^2x^2 + fx^2y^2) + 12xyz(lx + my + nz).$$

Then $\overline{123}^4$ is

$$abc - 4(ab_3c_2 + bc_1a_3 + ca_2b_1) + 3(ad^2 + be^2 + cf^2) + 4(a_2b_3c_1 \\ + a_3b_1c_2) - 12(a_2nd + a_3md + b_1ne + b_3le + c_1mf + c_2lf) \\ + 12(lb_1c_1 + mc_2a_2 + na_3b_3) + 12(dl^2 + em^2 + fn^2) + 6def \\ - 12lmn.$$

115. We can express in the same manner functions containing contragredient variables; for if α, β, γ be any variables contragredient to x, y, z, and therefore cogredient with $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, it follows, as before, that the determinant

$$\alpha\left(\frac{d}{dy_1}\frac{d}{dz_2} - \frac{d}{dy_2}\frac{d}{dz_1}\right) + \beta\left(\frac{d}{dz_1}\frac{d}{dx_2} - \frac{d}{dz_2}\frac{d}{dx_1}\right) + \gamma\left(\frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}\right)$$

(which we shall denote by the abbreviation $\overline{\alpha 12}$) is an invariantive symbol of operation. Thus, if U_1, U_2 be two different quadrics, $\overline{\alpha 12}^2$ expanded is

$$\alpha^2(b'c'' + b''c' - 2d'd'') + \beta^2(c'a'' + c''a' - 2e'e'') + \gamma^2(a'b'' + a''b' \\ - 2f'f'') + 2\beta\gamma(e'f'' + e''f' - a'd'' - a''d') + 2\gamma\alpha(f'd'' + f''d' \\ - b'e'' - b''e') + 2\alpha\beta(d'e'' + d''e' - e'f'' - e''f'),$$

which, when the two quadrics are identical, becomes the equation of the polar reciprocal of the quadric, as at p. 65.

In like manner, the curve contravariant to a quartic, which I have called S (" Higher Plane Curves," p. 101), may be written symbolically $\overline{\alpha 12}^4$, and the curve T in the same place may be

written $\overline{a12}^2\ \overline{a23}^2\ \overline{a31}^2$. In any of these we have only to replace the coefficient of any power of x, x^n by $\frac{d^n}{dx^n}$ to obtain a symbol which will yield a mixed concomitant when applied to a quantic of higher dimensions. Thus $\overline{a12}^2$ is

$$a^2\left\{\frac{d^2U}{dy^2}\frac{d^2U}{dz^2}-\left(\frac{d^2U}{dydz}\right)^2\right\}+\&c.,$$

which, when applied to a quadric, is a contravariant, but, when applied to a quantic of higher order, contains both x, y, z, as well as the contragredient a, β, γ, and, therefore, is a mixed concomitant.

In general, if we have the symbolical expression for any invariant of a binary quantic, we have only to prefix a contravariant symbol a to every term, when we shall have a contravariant of a ternary quantic of the same order. And in particular it can be proved that if we take the symbolical expression for the discriminant of a binary quantic, and prefix in this manner a contravariant symbol to each term, we shall have the expression for the polar reciprocal of a ternary quantic.

Thus, the symbol for the discriminant of a binary cubic is $\overline{12}^2.\overline{34}^2.\overline{13}.\overline{24}$, and the polar reciprocal of a ternary cubic is $\overline{a12}^2.\overline{a34}^2.\overline{a13}.\overline{a24}$, which is obviously of the sixth order in the variables a, β, γ, and of the fourth in the coefficients.

116. If in any contravariant we substitute $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ for a, β, γ, and operate on U, we get a covariant (Art. 94); and the symbol for this covariant is got from that for the contravariant by writing a new figure instead of a. Thus from $\overline{a23}^2$ is got $\overline{123}^2$, from $\overline{a23}.\overline{a24}$, is got $\overline{123}.\overline{124}$ &c. Conversely, if in the symbol for any invariant we replace any figure by a contravariant symbol a, we get the evectant of that invariant. Thus, if

$$\overline{123}.\overline{124}.\overline{234}.\overline{314}$$

be an invariant of a cubic, the evectant of that invariant is

$$\overline{123}.\overline{a12}.\overline{a23}.\overline{a31}.$$

In the case of a binary quantic, this rule assumes a simpler form; for if we substitute a contravariant symbol for 1 in $\overline{12}$, it becomes, when written at full length, $\xi \frac{d}{dy} - \eta \frac{d}{dx}$, but since ξ and η are cogredient with $-y$ and x, this may be written $x \frac{d}{dx} + y \frac{d}{dy}$, and may be suppressed altogether, since it only affects the result with a numerical multiplier. Hence, given the symbol for any invariant of a binary quantic, its evectant is got by omitting all the factors which contain any one figure. Thus,

$$\overline{12}^2 . \overline{34}^2 . \overline{13} . \overline{24}$$

being the discriminant of a cubic, its evectant, got by omitting the factors which contain 4, is $\overline{12}^2 . \overline{13}$.

If in a contravariant of any quantic we substitute $\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}$ for α, β, γ, we also get a covariant, and the symbol for it is obtained from that for the contravariant by writing a *different* new figure in place of every α. Thus, from $\overline{\alpha 34}^2$ we get $\overline{134} . \overline{234}$; and so on.

LESSON XII.

CANONICAL FORMS.

117. SINCE invariants and covariants retain their relations to each other, no matter how the quantic is linearly transformed, it is plain that when we wish to study these relations it is sufficient to do so by discussing the quantic in the simplest form to which it is possible to reduce it. This is only extending to quantics in general what the reader is familiar with in the case of ternary and quaternary quantics; since, when we wish to study the properties of a curve or surface, every geometer is familiar with the advantage of choosing such axes as shall reduce the equation of this curve or surface to its simplest form. The simplest form, then, to which a quantic can, without loss of

generality, be reduced is called the *canonical form* of the quantic. We can, by merely counting the constants, ascertain whether any proposed simple form is sufficiently general to be taken as the canonical form of a quantic, for if the proposed form does not, either explicitly or implicitly, contain as many constants as the given quantic in its most general form, it will not be possible always to reduce the general to the proposed form. Thus, we have proved (Art. 82) the possibility of reducing a binary cubic to the form $X^3 + Y^3$; and we might have seen this, *a priori*, from the consideration that the latter form, being equivalent to $(lx + my)^3 + (l'x + m'y)^3$ contains implicitly four constants, and therefore is as general as $(a, b, c, d \between x, y)^3$. So, in like manner, a ternary cubic in its most general form contains ten constants; but the form $X^3 + Y^3 + Z^3 + 6MXYZ$, contains also ten constants, since, in addition to the m which appears explicitly X, Y, Z, implicitly involve three constants each. This latter, then, may be taken as the canonical form of a ternary cubic, and, in fact, the most important advances that have been recently made in the theory of curves of the third degree are owing to the use of the equation in this simple and manageable form.

118. The quadratic function $(a, b, c \between x, y)^2$ can be reduced in an infinity of ways to the form $x^2 + y^2$, since the latter form implicitly contains four constants, and the former only three. In like manner the ternary quadric which contains six constants can be reduced in an infinity of ways to the form $x^2 + y^2 + z^2$, since this last contains implicitly nine constants; and in general a quadratic form in any number of variables can be reduced in an infinity of ways to a sum of squares. It is worth observing, however, that though a quadratic form can be reduced in an infinity of ways to a sum of squares, yet the number of positive and negative squares in this sum is fixed. Thus, if a binary quadric can be reduced to the form $x^2 + y^2$, it cannot also be reduced to the form $u^2 - v^2$, since we cannot have $x^2 + y^2$ identical with $u^2 - v^2$, the factors on the one side of the identity being imaginary, and those on the other being real. In like manner, for ternary quadrics we cannot have $x^2 + y^2 - z^2 = u^2 + v^2 + w^2$, since we should thus have $x^2 + y^2 = z^2 + u^2 + v^2 + w^2$, or, in other

words

$$x^2 + y^2 = z^2 + (lx + my + nz)^2 + (l'x + m'y + n'z)^2 + (l''x + m''y + n''z)^2,$$

and if we made x and $y = 0$, one side of the identity would vanish, and the other would reduce itself to the sum of four positive squares which could not be $= 0$. And the same argument applies in general.

119. We have shown (Art. 82) how to reduce a cubic to the sum of two cubes; and we shall now show, in general, that any binary quantic of odd degree $(2n-1)$ can be reduced to the sum of n powers of the $(2n-1)^{st}$ degree, a theorem due to Mr. Sylvester.* For the number of constants in any binary quantic is always one more than its degree, or, in the present case, $2n$; and we have the same number of constants if we take n terms of the form $(lx + my)^{2n-1}$. The actual transformation is performed by a method which is the generalization of that employed (Art. 82). For simplicity, we only apply it to the fifth degree, but the method is general. The problem then is to determine u, v, w, so that $(a, b, c, d, e, f \between x, y)^5$ may $= u^5 + v^5 + w^5$. Now we say that if we form the determinant

$$\begin{vmatrix} ax + by, & bx + cy, & cx + dy \\ bx + cy, & cx + dy, & dx + ey \\ cx + dy, & dx + ey, & ex + fy \end{vmatrix}$$

the three factors of this cubic will be u, v, w. For let

$$u = lx + my,\ v = l'x + m'y,\ w = l''x + m''y;$$

then, differentiating the identity

$$(a, b, c, d, e, f \between x, y)^5 = u^5 + v^5 + w^5,$$

four times successively with regard to x, and dividing by 120, we get

* It is to be observed that a quantic cannot *always* be reduced to its canonical form. The impossibility of the reduction indicates some singularity in the form of the quantic. Thus, if a cubic has a square factor, it cannot be reduced to the form $x^3 + y^3$, and its simplest form is x^2y.

$$ax + by = l^4u + l'^4v + l''^4w.$$

Similarly differentiating three times with regard to x, and once with regard to y,

$$bx + cy = l^3mu + l'^3m'v + l''^3m''w;$$

and so on.

The determinant, then, written above may be put into the form

$$\begin{vmatrix} l^4u + l'^4v + l''^4w, & l^3mu + l'^3m'v + l''^3m''w, & l^2m^2u + l'^2m'^2v + l''^2m''^2w \\ l^3mu + l'^3m'v + l''^3m''w, & l^2m^2u + l'^2m'^2v + l''^2m''^2w, & lm^3u + l'm'^3v + l''m''^3w \\ l^2m^2u + l'^2m'^2v + l''^2m''^2w, & lm^3u + l'm'^3v + l''m''^3w, & m^4u + m'^4v + m''^4w. \end{vmatrix}$$

Now it will be observed that the coefficients of u in every column are proportional to l^2, lm, m^2. Consequently, if we resolved this determinant, as in Art. 17, into partial determinants, every such determinant which contained two of the u columns would vanish as having two columns the same. And so, in like manner, would any which contained two v or two w columns. The determinant then will be uvw multipled by a numerical factor.*

When, then, the determinant written in the beginning of this Article has been found, by solving a cubic equation, to be the product of the factors $(x + \lambda y)(x + \mu y)(x + \nu y)$, we know that u, v, w, can only differ from these by numerical coefficients, and we may put

$$(a, b, c, d, e, f \between x, y)^5 = A(x + \lambda y)^5 + B(x + \mu y)^5 + C(x + \nu y)^5;$$

and then A, B, C are got from solving any of the systems of simple equations got by equating three coefficients on both sides of the above identity.

The determinant used in this Article is a covariant, which is called the canonizant of the given quantic.

120. The canonizant may be written in another, and perhaps simpler form, namely,

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* Viz. $(lm' - l'm)^2 (l'm'' - l''m')^2 (l''m - lm'')^2$.

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix}$$

This last is the form in which we should have been led to it if we had followed the course that naturally presented itself, and sought directly to determine the six quantities, A, B, C, λ, μ, ν, by solving the six equations got on comparison of coefficients of the identity last written in Art. 119. For the development of the solution in this form, to which we cannot afford the necessary space here, we refer to Mr. Sylvester's Paper ("Philosophical Magazine," November, 1851). Meanwhile, the identity of the determinant in this Article with that in the last has been shown by Mr. Cayley as follows. We have, by multiplication of determinants (Art. 17),

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix} \times \begin{vmatrix} 1, & 0, & 0, & 0 \\ x, & y, & 0, & 0 \\ 0, & x, & y, & 0 \\ 0, & 0, & x, & y \end{vmatrix}$$

$$= \begin{vmatrix} y^3, & 0 & , & 0 & , & 0 \\ 0, & ax+by, & & bx+cy, & & cx+dy \\ 0, & bx+cy, & & cx+dy, & & dx+ey \\ 0, & cx+dy, & & dx+ey, & & ex+fy \end{vmatrix}$$

which, dividing both sides of the equation by y^3, gives the identity required.

121. We have still to mention another way of forming the canonizant. Let this sought covariant be $(A, B, C, D\between x, y)^3$ where we want to determine A, B, C, D; then (Art. 76) $\left(A, B, C, D\between \frac{d}{dy}, -\frac{d}{dx}\right)^3$ will also yield a covariant. But if this operation is applied to $(x+\lambda y)^n$ where $x+\lambda y$ is a factor in $(A, B, C, D\between x, y)^3$, the result must vanish since one of the factors in the operating symbol is $\frac{d}{dy} - \lambda\frac{d}{dx}$. Since, then, the

given quantic is by hypothesis the sum of three terms of the form $(x + \lambda y)^5$, the result of applying to the given quantic the operating symbol just written must vanish. Thus, then, we have

$$A(d, e, f\between x, y)^2 - B(c, d, e\between x, y)^2 + C(b, c, d\between x, y)^2 - D(a, b, c\between x, y)^2 = 0,$$

or, equating separately to 0 the coefficients of x^2, xy, y^2, we have

$$\begin{aligned} Ad - Bc + Cb - Da &= 0 \\ Ae - Bd + Cc - Db &= 0 \\ Af - Be + Cd - Dc &= 0 \end{aligned}$$

whence (Art. 24) A is proportional to the determinant got by suppressing the column A or $\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}$ and so for B, C, D, which values give for the canonizant the form stated in the last Article.

122. We proceed now to quantics of even degree $(2n)$. Since this quantic contains $2n + 1$ terms, if we equate it to a sum of n powers of the degree $2n$, we have one equation more to satisfy than we have constants at our disposal. On the other hand, if we add another $2n^{th}$ power, we have one constant too many, and the quantic can be reduced to this form in an infinity of ways. It is easy, however, to determine the condition that the given quantic should be reducible to the sum of n, $2n^{th}$ powers. Thus, for example, the conditions that a quartic should be reducible to the sum of two fourth powers, and that a sextic should be reducible to the sum of three sixth powers, are respectively the determinants,

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0 \qquad \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} = 0,$$

and so on. For in the case of the quartic the constituents of

the determinant are the several fourth differentials of the quantic, and, expressing these in terms of u and v precisely as in Art. 119, it is easy to see that the determinant must vanish, when the quartic can be reduced to the form $u^4 + v^4$. Similarly for the rest. This determinant expanded in the case of the quartic is the invariant already noticed (see Art. 86),

$$ace + 2bcd - ad^2 - eb^2 - c^3.$$

123. When this condition is not fulfilled, the quantic is reduced to the sum of n powers, together with an additional term. Thus, the canonical form for a quartic is $u^4 + v^4 + 6\lambda u^2v^2$. We shall commence with the reduction of the general quartic to its canonical form; the method which we shall use is not the easiest for this case, but is that which shows most readily how the reduction is to be effected in general. Let the product, then, of u, v, which we seek to determine, be $(A, B, C\between x, y)^2$, and let us operate with $\left(A, B, C\between\frac{d}{dy}, -\frac{d}{dx}\right)^2$ on both sides of the identity $(a, b, c, d, e\between x, y)^4 = u^4 + v^4 + 6\lambda u^2v^2$.

Now, as before, this operation performed on u^4 and on v^4 will vanish, and when performed on $6\lambda u^2v^2$, it will be found to give $12\lambda' uv$ where $\lambda' = 2(4AC - B^2)\lambda$. Equating then the coefficients of x^2, xy, and y^2 on both sides after performing the operation, we get the three equations

$$\begin{aligned} Ac - 2Bb + Ca &= \lambda' A \\ Ad - 2Bc + Cb &= \lambda' B \\ Ae - 2Bd + Cc &= \lambda' C \end{aligned}$$

whence eliminating A, B, C, we have to determine λ', the determinant

$$\begin{vmatrix} a, & b, & c-\lambda' \\ b, & c+\tfrac{1}{2}\lambda', & d \\ c-\lambda', & d, & e \end{vmatrix} = 0,$$

which expanded is the cubic

$$\lambda'^3 - \lambda'(ae - 4bd + 3c^2) - 2(ace + 2bcd - ad^2 - eb^2 - c^3) = 0,^*$$

* N. B—The discriminant of this cubic is the same as that of the quartic.

the coefficients of which are invariants. Thus, then, we have a striking difference in the reduction of binary quantics to their canonical form, between the cases where the degree is odd and where it is even. In the former case, the reduction is unique, and the system u, v, w, &c. can be determined in but one way. When u is of even degree, however, more systems than one can be found to solve the problem. Thus, in the present instance, a quartic can be reduced in three ways to the canonical form, and if we take for λ' any of the roots of the above cubic, its value substituted in the preceding system of equations enables us to determine A, B, C.

124. If now we proceed to the investigation of the reduction of the quantic $(a_0, a_1, a_2 \ldots \between x, y)^{2n}$, the most natural canonical form to assume would be $u^{2n} + v^{2n} + w^{2n} + \&c. + \lambda u^2v^2w^2$ &c., there being n quantities u, v, w, &c. But the actual reduction to this form is attended with difficulties which have not been overcome, except for the cases $u = 2$ and $u = 4$. But the method used in the last Article can be applied if we take for the canonical form $u^{2n} + v^{2n} + \&c. + \lambda Vuvw$ &c., where, if

$$uvw \text{ \&c.} = (A_0, A_1, A_2 \ldots \between x, y)^n,$$

V is a covariant of this latter function such that when $Vuvw$ &c. is operated on by $\left(A_0, A_1 \ldots \between \frac{d}{dy}, -\frac{d}{dx}\right)^n$, the result is proportional to the product uvw &c. Suppose for the moment that we had found a function V to fulfil this condition, then, proceeding exactly as in the last Article, and operating with the differential symbol last written on the identity got by equating the quantic to its canonical form, we get the system of equations

$$A_0a_n - nA_1a_{n+1} + \frac{n(n-1)}{1 \cdot 2} A_2a_{n+2} - \&c. = \lambda A_0$$

$$A_0a_{n-1} - nA_1a_n + \frac{n(n-1)}{1 \cdot 2} A_2a_{n+1} - \&c. = \lambda A_1$$

$$A_0a_{n-2} - nA_1a_{n-1} + \frac{n(n-1)}{1 \cdot 2} A_2a_n - \&c. = \lambda A_2, \text{ \&c.},$$

whence, eliminating A_0, A_1, A_2, &c., we get the determinant

$$\begin{vmatrix} a_n - \lambda, & a_{n+1}, & a_{n+2}, \dots\dots\dots\dots & a_{2n} \\ a_{n-1}, & a_n + \dfrac{1}{n}\lambda, & a_{n+1}, \dots\dots\dots\dots & a_{2n-1} \\ a_{n-2}, & a_{n-1}, & a_n - \dfrac{1\,.\,2}{n(n-1)}\lambda, \;\dots & a_{2n-2} \\ \dots & \dots & \dots & \dots \\ a_0, & a_1, & a_2 \dots\dots\dots\dots & a_n \mp \lambda \end{vmatrix}^*$$

and having found λ by equating to 0 this determinant expanded (a remarkable equation, all the coefficients of which will be invariants), the equations last written enable us to determine the values of A_0, A_1, &c., corresponding to any of the $n+1$ values of λ.

125. To apply this to the case of the sextic, the canonical form here is $u^6 + v^6 + w^6 + Vuvw$, where, if uvw be

$$(A_0, A_1, A_2, A_3 \between x, y)^3,$$

V is the evectant of the discriminant of this last quantic, and whose value is written at full length (Ex. p. 62). Now it will afford an excellent example of the use of canonical forms if we show that in any cubic the result of the operation

$$(a_0, a_1, a_2, a_3 \between \frac{d}{dy}, -\frac{d}{dx}\Big),$$

performed on the product of the cubic and the evectant just mentioned, will be proportional to the cubic itself. For it is sufficient to prove this, for the case when the cubic is reduced to

* The determinant above written may be otherwise obtained as follows. Let x', y' be cogredient to x, y, and let us form the function

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^n U + \lambda\,(xy' - yx')^n,$$

which (Arts. 79, 87), we have proved to be linearly transformed into a function of similar form. Take the $n+1$ coefficients of the several powers x^n, $x^{n-1}y$, &c., and from these eliminate linearly the $n+1$ quantities x'^n, x'^{n-1} &c., and we obtain the determinant in question.

the canonical form $x^3 + y^3$, in which case the evectant will be $x^3 - y^3$ as appears at once by putting $b = c = 0$, and $a = d = 1$ in the value given p. 62. The product, then, of cubic and evectant will be $x^6 - y^6$, which, if operated on by $\frac{d^3}{dy^3} - \frac{d^3}{dx^3}$, gives a result manifestly proportional to $x^3 + y^3$. And the theorem now proved being independent of linear transformation, if true for any form of the cubic, is true in general. The canonical form, then, being assumed as above, we proceed exactly as in the last Article, and we solve for λ from the equation

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3 - \lambda \\ a_1, & a_2, & a_3 + \frac{1}{3}\lambda, & a_4 \\ a_2, & a_3 - \frac{1}{3}\lambda, & a_4, & a_5 \\ a_3 + \lambda, & a_4, & a_5, & a_6 \end{vmatrix} = 0,$$

which, when expanded, will be found to contain only even powers of λ. If we suppose uvw reduced as above to its canonical form $x^3 + y^3$, the three factors of which are

$$x + y,\ x + \omega y,\ x + \omega^2 y,$$

where ω is a cube root of unity, then it is evident from the above that the corresponding canonical form for the sextic is

$$A(x + y)^6 + B(x + \omega y)^6 + C(x + \omega^2 y)^6 + D(x^6 - y^6).$$

It can be proved (see next Lesson) that if u, v, w be the factors of the cubic, then the factors of the evectant used above are $u - v$, $v - w$, $w - u$, so that the canonical form of the sextic may also be written

$$u^6 + v^6 + w^6 + \lambda uvw(u - v)(v - w)(w - u).$$

126. In the case of the octavic the canonical form is

$$u^8 + v^8 + w^8 + z^8 + \lambda u^2v^2w^2z^2,$$

for if we operate on $u^2v^2w^2z^2$ with a symbol formed according to the same method as in the preceding Articles, the result will be proportional to $uvwz$. We reserve the proof of this as an exer-

cise in the Hyperdeterminant Calculus in a subsequent Lesson. The canonical forms for higher binary quantics have not been formed.

The canonical form for a ternary cubic is, as we have already stated, $x^3 + y^3 + z^3 + 6mxyz$, and for a quaternary cubic, as Mr. Sylvester has pointed out, $x^3 + y^3 + z^3 + u^3 + v^3$. Higher canonical forms have not been determined.

LESSON XIII.

APPLICATIONS TO BINARY QUANTICS.

127. Having now explained the most essential parts of the general theory, we wish to give some additional details as to the forms of invariants and covariants which most frequently occur in practice. We shall give the applications of the theory to binary quantics in this, and to ternary in the next Lesson.

To commence with the single quadratic form $(a, b, c \between x, y)^2$, it has got but one independent invariant, viz., the discriminant $(ac - b^2)$. Any other invariant must be a power of this, $(ac - b^2)^m$. We have already showed (Art. 112) that it follows by Hermite's Law of Reciprocity that only quantics of even degree can have quadratic invariants whose symbol is $\overline{12}^{2m}$. If we make $y = 1$ in the quantic, it may be taken to represent geometrically a system of two points on the axes of x, and the vanishing of the discriminant expresses the condition that these points should coincide.

128. We proceed next to a system of two quadrics

$$(a, b, c \between x, y)^2, \ (a', b', c' \between x, y)^2,$$

and we have seen that they have the invariant $\overline{12}^2$ or $ac' + a'c - 2bb'$. When each quantic is taken to represent a pair of points as in the last Article, then the invariant just written expresses the condition (see "Conics," p. 287) that the four points should form a harmonic system, the two points represented by the same

quantic being conjugate to each other. We have also proved (see "Conics," p. 288) that the covariant $\overline{12}$ (or the Jacobian of the system of quadrics) represents geometrically the foci of the system in involution determined by the four points.

We have proved (p. 20) that the eliminant of the system of two quadrics is

$$(ac' - ca')^2 + 4(ba' - b'a)(bc' - b'c),$$

but the same eliminant can be also represented in a form due to Dr. Boole, which has all its factors invariants, viz.

$$(ac + ca' - 2bb')^2 = 4(ac - b^2)(a'c' - b'^2).$$

We might have obtained the result in this form as follows:—Take the quantic $(a, b, c \between x, y)^2 + \lambda(a', b', c' \between x, y)^2$ and form its discriminant, viz. $(a + \lambda a')(c + \lambda c') - (b + \lambda b')^2$. Arranging this latter quantity according to the powers of λ, and forming its discriminant, we get, in the form above written, an invariant of the system of quantics which is in this case their eliminant. The method, however, is not applicable to finding the eliminant of any two quantics. If we form the discriminant of

$$ax^n + \&c. + \lambda(a'x^n + \&c.),$$

and then form the discriminant of this again with respect to λ, we get an invariant of the two quantics, which will indeed contain their eliminant as a factor, but which, except when $n = 2$, will be of much higher degree, and will, therefore, have other factors besides.

129. We come next to the concomitants of the cubic

$$(a, b, c, d \between x, y)^3.$$

It has but one invariant, viz., the discriminant

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd.$$

The Hessian $\overline{12}^2$ is

$$(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2,$$

which has the same discriminant as the cubic itself, see Art. 85. If the roots of the cubic be α, β, γ, then the Hessian is

$\Sigma(x-\alpha)^2(\beta-\gamma)^2$ (see Art. 75). The cubic covariant $\overline{12^2}.\overline{13}$, or the evectant of the discriminant, is (see p. 62),

$$(a^2d + 2b^3 - 3abc)x^3 + 3(b^2c + abd - 2ac^2)x^2y \\ + 3(2b^2d - bc^2 - acd)xy^2 + (3bcd - ad^2 - 2c^3)y^3.$$

This cubic may be geometrically represented as follows:—If we take the three points represented by the cubic itself, and take the fourth harmonic of each with respect to the other two, we get three new points which will be the geometrical representation of the covariant in question. This theorem is suggested by its being evident on inspection that if the given cubic take the form $xy(x+y)$, then $x-y$ will be a factor in the covariant, as appears by making $a=d=0$, $b=c=1$ in its equation. But $x+y$, $x-y$ are harmonic conjugates with respect to x and y. And harmonic relations (see "Higher Plane Curves," p. 226) are unaltered by linear transformation, so that if they have been proved to exist in one case, they exist in general. This leads us to the expression for the factors of the covariant in terms of the roots of the given cubic: for if δ be the distance from the origin of the point conjugate to α with respect to β and γ; solving for δ from the equation $\dfrac{2}{\alpha-\delta} = \dfrac{1}{\alpha-\beta} + \dfrac{1}{\alpha-\gamma}$ we get $\delta = \dfrac{\alpha\beta + \alpha\gamma - 2\beta\gamma}{2\alpha - \beta - \gamma}$, whence the covariant must be

$$\{(2\alpha-\beta-\gamma)x + (2\beta\gamma - \alpha\beta - \alpha\gamma)y\}\{(2\beta-\alpha-\gamma)x \\ + (2\gamma\alpha - \beta\gamma - \beta\alpha)y\}\{(2\gamma-\alpha-\beta)x + (2\alpha\beta - \gamma\alpha - \gamma\beta)y\},$$

as may be verified by actual multiplication and substitution in terms of the coefficients of the equation.

130. When we wish to establish any relation between the preceding covariants andinvariants, we use the canonical forms, which are, for $U = ax^3 + dy^3$, the discriminant $D = a^2d^2$; the Hessian $H = adxy$; and the cubicovariant, $J = ad(ax^3 - dy^3)$. Thus we can prove that the discriminant of J is the cube of the discriminant of U, for the discriminant of J in its canonical form

is a^6d^6. Again, we can easily prove in the same way a relation due to Mr. Cayley, viz.,

$$J^2 - DU^2 = -4H^3.$$

Mr. Cayley has used this equation to solve the cubic U, or, in other words, to resolve it into its linear factors. For since $J^2 - DU^2$ is a perfect cube, we are led to infer that the factors $J \pm U\sqrt{D}$ will also be perfect cubes, and in fact, the canonical form shows that they are $2a^2dx^3$ and $2ad^2y^3$. Now since $xa^{\frac{1}{3}} + yd^{\frac{1}{3}}$ is one of the factors of the canonical form, it immediately follows that the factor in general is proportional to

$$(U\sqrt{D} + J)^{\frac{1}{3}} + (U\sqrt{D} - J)^{\frac{1}{3}},$$

a linear function which evidently vanishes on the supposition $U = 0$.

Ex. Let us take the same example as at p. 59, viz., $U = 4x^3 + 9x^2 + 18x + 17$. Here we have $D = 1600$, $J = 110x^3 - 90x^2y - 630xy^2 - 670y^3$, whence

$$U + J\sqrt{D} = 10\,(3x + y)^3\,; \quad U - J\sqrt{D} = 50\,(x + 3y)^3,$$

and the factors are $3x + y + (x + 3y)\sqrt[3]{5}$.

131. We come next to the quartic, which, as we have seen (Arts. 77, 86), has the two invariants

$$S = ae - 4bd + 3c^2 \text{ and } T = ace + 2bcd - ad^2 - eb^2 - c^3.$$

The canonical form is $x^4 + 6mx^2y^2 + y^4$, and for this form these invariants are $S = 1 + 3m^2$, $T = m - m^3$.

These invariants, expressed as symmetric functions of the roots, are

$$S = \Sigma(\alpha - \beta)^2(\gamma - \delta)^2, \quad T = \Sigma(\alpha - \beta)^2(\gamma - \delta)^2(\alpha - \gamma)(\beta - \delta),$$

or, more conveniently,

$$= \{(\alpha - \beta)(\gamma - \delta) - (\alpha - \gamma)(\delta - \beta)\}\{(\alpha - \gamma)(\delta - \beta) \\ - (\alpha - \delta)(\beta - \gamma)\}\{(\alpha - \delta)(\beta - \gamma) - (\alpha - \beta)(\gamma - \delta)\}.$$

In the latter form it is easy to see that $T = 0$ is the condition that the four points represented by the quartic should form a

harmonic system (" Higher Plane Curves," p. 192). If M be the modulus of transformation, it follows from general principles that in transformation S becomes M^4S' and T becomes M^6T': hence the ratio $S^3 : T^2$ is *absolutely* unaltered by transformation. Every other invariant can be expressed as a rational function of these two.* Thus, for instance, the discriminant is easily expressed in terms of S and T by the help of the canonical form; for, treating the latter like a quadratic, its discriminant is $(1 - 9m^2)^2$ which $= S^3 - 27T^2$.

132. This relation between the invariants of a quartic may be used to prove the relation between the covariants of a cubic given (Art. 130), for if we take a cubic and multiply by $xy' - yx'$, the coefficients of the quartic so formed will be

$$ay', \tfrac{1}{4}(3by' - ax'), \tfrac{1}{2}(ay' - bx'), \tfrac{1}{4}(dy' - 3cx'), - dx',$$

and the invariants of this quartic will be covariants of the cubic The discriminant is proportional to U^2D (see Art. 67), the invariant $S = \frac{3}{4}H$, $T = \frac{1}{16}J$, whence we derive the relation between these covariants already given.

133. The Hessian of the quartic is

$$(ac - b^2)x^4 + 2(ad - bc)x^3y + (ae + 2bd - 3c^2)x^2y^2 + 2(bc - cd)xy^3 + (ce - d^2)y^4,$$

which, for the canonical form, is $m(x^4 + y^4) + (1 - 3m^2)x^2y^2$.

It has been already remarked (p. 60) that when the quartic is resolvable into two square factors, the coefficients of the Hessian are proportional to those of the quartic.

The reduction of the quartic to its canonical form is easily effected by the values given for S and T, for since $S = 1 + 3m^2$, $T = m - m^3$, we have $4m^3 - mS + T = 0$, which determines m. When this has been found, we can get the x and y, which occur in the canonical form, from the equations

* For the proof of this we refer to Mr. Sylvester's Paper ("Philosophical Magazine," April, 1853), or to Mr. Cayley's second memoir on quantics in the "Philosophical Transactions," already referred to.

$$x^4 + y^4 + 6mx^2y^2 = U, \quad m(x^4 + y^4) + (1 - 3m^2)x^2y^2 = H,$$

whence $x^2y^2 = \dfrac{H - mU}{1 - 9m^2}$, so that by taking the square root of this quantity, and solving the quadratic so found, the factors x and y are determined. Mr. Cayley, however, has given the solution of the biquadratic under a more elegant form. Let m_1, m_2, m_3 be the three roots of the equation above written, which determines m, then it has been already proved that

$$H - m_1U, \; H - m_2U, \; H - m_3U$$

are perfect squares, and the square roots will be, of course, of the second degree in x and y. Now, Mr. Cayley has found that

$$(m_2 - m_3)(H - m_1U)^{\frac{1}{2}} + (m_3 - m_1)(H - m_2U)^{\frac{1}{2}} \\ + (m_1 - m_2)(H - m_3U)^{\frac{1}{2}}$$

will be also a perfect square, and that the linear factor which is its root is one of the factors of the quartic. It is only necessary to prove that this quantity is a perfect square, for it evidently vanishes on the supposition $U = 0$. Now, if we solve the equation $4z^3 - z(1 - 3m^2) + m - m^3 = 0$, the three roots are easily found to be $m_1 = m$, $m_2 = -\dfrac{m+1}{2}$, $m_3 = -\dfrac{m-1}{2}$. Whence

$$H - m_1U = (1 - 9m^2)x^2y^2, \qquad H - m_3U = \frac{3m - 1}{2}(x^2 - y^2)^2$$

$$H - m_2U = \frac{3m + 1}{2}(x^2 + y^2)^2.$$

Now, in order that any quantity of the form

$$\alpha(x^2 + y^2) + \beta(x^2 - y^2) + \gamma xy$$

should be a perfect square, we must obviously have $\gamma^2 = 4(\alpha^2 - \beta^2)$ and on putting for α, β, and γ their values, it is readily seen that this condition is fulfilled.

134. Quartics have another covariant which is the cubicovariant J of the first emanant, or symbolically is $\overline{12}^2.\overline{13}$. It is of the third degree in the coefficients, and the sixth in the vari-

ables. For the canonical form this covariant is $(1-9m^2)xy(x^4-y^4)$. The reader might have expected that a quartic like a cubic would have had a quadratic covariant, whose factors would have been the x and y of the canonical form. But as the canonical form is threefold (because m is determined by a cubic equation), the form for determining the factors of this canonical form rises to the sixth degree, and is this very covariant J. Geometrically, the matter may also be stated as follows. Four points on a line determine three different systems in involution (because either the point B, C, or D may be taken as conjugate to A), and the foci of these three systems are determined by the covariant J.

Since by the last Article the square of the product of one set of x and y of the canonical form is determined by $H-mU$ we have J^2 proportional to $(H-m_1U)(H-m_2U)(H-m_3U)$, or (from the equation which determines m), to $4H^3 - SHU^2 + TU^3$.

135. If we form the Hessian of the Hessian of a quartic, it will be of the form $x^4+y^4+6m'x^2y^2$, and, therefore, can be expressed under the form $U+\lambda H$. The actual values are easily got by the canonical form. We have to form the Hessian of

$$m(x^4+y^4)+(1-3m^2)x^2y^2$$

which will be

$$2m(1-3m^2)(x^4+y^4)-(1-18m^2+9m^4)x^2y^2;$$

or, putting for x^4+y^4, $\dfrac{(1-3m^2)U-6mH}{1-9m^2}$, and for x^2y^2, $\dfrac{H-mU}{1-9m^2}$, we get for the Hessian of the Hessian

$$3TU-SH.$$

We can prove, in precisely the same way, that the Hessian of $U+\lambda H$ (or any other covariant of the fourth order) is of the form $U+\lambda' H$, and can determine the value of λ' in terms of λ, S, and T. Thus, the discriminant of the first emanant is a covariant of the fourth order, and it is easily proved in the same way to be equal to $SH-TU$. So again, if U be *any* quantic, α, β, γ, δ its third differentials, then

$$\alpha^2\delta^2+4\alpha\gamma^3+4\delta\beta^3-3\beta^2\gamma^2-6\alpha\beta\gamma\delta$$

is a covariant. Now if a, b, c, d, e be the fourth differentials, we can, by the theorem of homogeneous functions, substitute $(n-3)\,a = ax + by$, &c., and thus express the above covariant in terms of the fourth differentials; and as the form of the result must be exactly the same as for the case of a quartic, it is evident that this covariant is reducible to the form $SH + \lambda TU$ where S and T are the covariants got by substituting fourth differentials for the coefficients, in the S and T of a quartic. In like manner, it is proved that the Hessian of the Hessian of *any* binary quantic can be expressed under the form $SH + \lambda TU$, a theorem of which we shall give another proof in a subsequent Lesson on Hyperdeterminants.

136. We shall only briefly indicate the principal results which have been obtained with respect to quintics. Mr. Sylvester first discussed them under the form $ax^5 + by^5 + cz^5$, where $x + y + z = 0$ (see Art. 119). He showed that they have three fundamental invariants of the fourth (see Art. 85), eighth, and twelfth degrees respectively, which in terms of the canonical form are

$$A = a^2b^2 + b^2c^2 + c^2a^2 - 2abc\,(a + b + c)$$
$$B = a^2b^2c^2\,(ab + bc + ca); \quad C = a^4b^4c^4.$$

The discriminant is $A^2 - 128B$. Quintics have, of course, innumerable other invariants of the degree $4m$, expressible as rational functions of the invariants A, B, C, and it was for some time thought that they could have invariants of no other form. Subsequently, M. Hermite discovered the existence of an invariant of the eighteenth degree. In fact, if we express the following invariant function of the thirty-sixth degree, by the help of the canonical form,

$$A\,(AC - B^2)^2 + 8B^3C - 72ABC^2 - 432C^3,$$

it will be found to be a perfect square, the root being the invariant of the eighteenth degree in question.

There is a remarkable difference between this invariant and any others which we have yet formed. If M be the modulus of transformation, any invariant I becomes by transformation M^nI,

and now in all the cases which have occurred previously, n was an even number. Now, if we change x into y and y into x, this is a transformation whose modulus is -1, and in all the former cases the invariant would be absolutely unaltered by such a transformation; but in the case of Hermite's (which may be called a skew invariant) n is odd, and changing x into y and y into x will alter the sign of the invariant. Consequently, one term, for example, in this invariant being $a^7d^5f^6$, the corresponding term will have an opposite sign and be $-a^6c^5f^7$. The actual values of the several invariants of a quintic will be found calculated at length in Mr. Cayley's Memoirs on Quantics.

137. M. Hermite arrived at his invariant by studying the quintic under a new canonical form. We have seen (Art. 83) that every quantic of odd degree has a quadratic covariant, and also (Art. 82) that a cubic is changed into its canonical form by taking for the new x and y the two factors of this covariant. M. Hermite applies exactly the same transformation to a quintic, and supposes it written in such a way that its quadratic covariant shall reduce to the single term xy; in other words, he supposes the coefficients connected by the relations $ae - 4bd + 3c^2 = 0$, $bf - 4ce + 3d^2 = 0$. Now, we know that we may substitute in any covariant $\frac{d}{dx}$, $-\frac{d}{dy}$ for y and x; and the advantage of the canonical form here chosen is that we see that, since xy is now a covariant, $\frac{d^2}{dxdy}$ is an invariantive operating symbol, the simplicity of which enables us with great ease to deduce one covariant from another. Thus, we see immediately, that the quintic $(a, b, c, d, e, f\between x, y)^5$ possesses a covariant of the first degree, which, for Hermite's canonical form, is $cx + dy$; and so on. M. Hermite's memoir will be found in the "Cambridge and Dublin Mathematical Journal," vol. ix., p. 172.

LESSON XIV.

APPLICATIONS TO TERNARY QUANTICS.

138. We shall in this Lesson enumerate the more important of the invariants and covariants of ternary quadrics, and give some illustrations of the use to be made of them in the theory of curves. We commence with the quadric

$$(a, b, c, d, e, f \between x, y, z)^2,$$

which has but one invariant, namely, its discriminant

$$abc + 2def - ad^2 - be^2 - cf^2,$$

and one contravariant, namely, the evectant of the discriminant (Art. 96). These can be both written in the form of determinants, viz.:—

$$\begin{vmatrix} a, & f, & e \\ f, & b, & d \\ e, & d, & c \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a, & f, & e, & \xi \\ f, & b, & d, & \eta \\ e, & d, & c, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix}$$

If we were considering a quantic of any degree, and a, b, &c. denote second differentials, the first of these would be the Hessian, and the second a mixed concomitant, which, from the form of the determinant, has been called the bordered Hessian.

The quadric can be reduced in an infinity of ways to the canonical form $x^2 + y^2 + z^2$; in which case x, y, z form what we have called a self-conjugate triangle; that is to say, three lines such that each is the polar of the intersection of the other two (see "Conics," pp. 232, 321).

139. Let us go on now to consider the case of two quadrics U, V. Then we get a set of invariants of the system by forming the discriminant of $U + \lambda V$, when the coefficients of the several powers of λ must obviously be invariants. Let this discriminant be $\Delta + \lambda\Theta + \lambda^2\Theta' + \lambda^3\Delta'$, then Δ and Δ' are plainly the discriminants of U and V, while

$$\Theta = \left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc} + d'\frac{d}{dd} + e'\frac{d}{de} + f'\frac{d}{df}\right)\Delta,$$

or, at full length

$$= a'(bc - d^2) + b'(ca - e^2) + c'(ab - f^2)$$
$$+ 2d'(ef - ad) + 2e'(fd - be) + 2f'(de - cf).$$

All permanent relations between the two conics can be expressed in terms of these invariants. Thus (see "Conics," p. 252), the condition that the conics should touch is the discriminant with respect to λ of the discriminant of $U + \lambda V$ written above.

So again, $\Theta = 0$ expresses the condition that the second conic passes through the vertices of a self-conjugate triangle of the first. For if this were the case, it would be possible to transform simultaneously, the first to the form $x^2 + y^2 + z^2$, the second to the form $2d'yz + 2e'zx + 2f'xy$. Now, the supposition $a' = b' = c' = 0$ reduces Θ to

$$d'(ef - ad) + e'(fd - be) + f'(de - cf),$$

which vanishes on the supposition $d = e = f = 0$. And when it has been proved that Θ vanishes in any case, since it is an invariant, it must vanish for any form into which the same equations can be transformed.

In like manner, let it be required to find the condition that it may be possible that a triangle should be inscribed in one and circumscribed to the other. Let x, y, z be the sides of such a triangle: then it will be possible to transform the conics so that the one shall take the form $2dyz + 2ezx + 2fxy$, and the other

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy.$$

And whatever relations we can establish between the invariants of these forms must hold generally. But these invariants are

$$\Delta = 2def;\quad \Theta = -(d + e + f)^2;\quad \Theta' = 4(d + e + f);\quad \Delta' = -4.$$

These manifestly fulfil the relation $\Theta'^2 = 4\Theta\Delta'$, which is the required condition. The reader may ask why we did not write down the relation $\Delta' = -4$ as one which must be true in general.

He will remember that an invariant is in general not absolutely unaltered by transformation, but is multiplied by a power of the modulus, and consequently, the equation $\Delta' = -4$, would, on transformation, cease to be true. But any homogeneous relation between Δ, Δ', Θ, Θ', if once true, will always be true, since every term is, on transformation, multiplied by the same factor.

140. We pass on now to speak of covariants and contravariants of the system of equations, and commence with the latter. The reciprocal of the system $U + \lambda V$ is $\Sigma + \lambda\Phi + \lambda^2\Sigma'$ (see "Conics," p. 268), where Σ, Σ' are the reciprocals of U and V, and Φ is

$$(bc' + b'c - 2dd')\xi^2 + (ca' + c'a - 2ee')\eta^2 + (ab' + a'b - 2ff')\zeta^2$$
$$+ 2(ef' + e'f - ad' - a'd)\eta\zeta + 2(fd' + f'd - be' - b'e)\zeta\xi$$
$$+ 2(de' + d'e - cf' - c'f)\xi\eta.$$

$\Phi = 0$ expresses the condition that $x\xi + y\eta + z\zeta$ should be cut harmonically by the two conics. The equation $\Phi^2 = 4\Sigma\Sigma'$ expresses the condition that the same line should pass through one of the four points of intersection of the two conics, and may be called the tangential equation of these points.

Suppose now, that in Φ we write for a, b, c, &c. the corresponding coefficients of the reciprocal conics, and x, y, z for ξ, η, ζ, we get a quadric F *co*variant to the given conics. It denotes the locus of a point whence tangents to the two conics form a harmonic pencil; and the equation of common tangents to the two is $F^2 = 4\Delta\Delta' UV$ (see "Conics," pp. 268, 288).

All conics covariant to the given conics can be expressed in terms of U, V, and F. Thus, for example, we know that if we substitute $\frac{dU}{dx}$ &c. for ξ, η, ζ in any contravariant, we shall get a covariant. Let us then examine the result of making this substitution in the reciprocal of V. The geometrical meaning will be the locus of the poles with respect to U of the tangents to V. In these, and similar questions, it is convenient to use the canonical form. Thus, we know (see "Conics," p. 267) that there are three lines whose poles are the same with regard

to both conics, and if these be taken for x, y, z, the conics would assume the form

$$U = ax^2 + by^2 + cz^2, \quad V = a'x^2 + b'y^2 + c'z^2;$$

whence we have

$$\Sigma = bc\xi^2 + ca\eta^2 + ab\zeta^2, \qquad \Sigma' = b'c'\xi^2 + c'a'\eta^2 + a'b'\zeta^2,$$
$$\Phi = (bc' + b'c)\xi^2 + (ca' + ac')\eta^2 + (ab' + ba')\zeta^2,$$
$$F = a'a'(bc' + b'c)x^2 + bb'(ca' + c'a)y^2 + cc'(ab' + ba')z^2.$$

We have also

$$\Delta = abc, \quad \Delta' = a'b'c', \quad \Theta = a'bc + b'ca + c'ab, \quad \Theta' = ab'c' + bc'a' + ca'b'.$$

If now we substitute in Σ', ax, by, cz, for ξ, η, ζ, we get

$$b'c'a^2x^2 + c'a'b^2y^2 + a'b'c^2z^2,$$

which the above values show to be equal to $\Theta'U - F$.

It is to be observed that theorems of this kind proved for conics are equally true for quantics of any order if the coefficients be understood to denote the corresponding second differentials; Δ and Δ' would then be the Hessians of the two quantics; and Θ, Θ' would be covariants. Thus, for example, if in the covariant $\Psi = \left\{\frac{d^2V}{dx^2}\frac{d^2V}{dy^2} - \left(\frac{d^2V}{dxdy}\right)^2\right\}\left(\frac{dU}{dz}\right)^2 + \&c.$ we express the first differentials of U in terms of the second, by the help of the theorem of homogeneous functions, we get a quantity of exactly similar form to that just discussed; and we get

$$(n-1)^2\Psi = n(n-1)\Theta'U - F,$$

or, again, if Ψ' be the covariant got from Ψ, by interchanging U and V we have

$$(n-1)^2\Psi - (n'-1)^2\Psi' = n(n-1)\Theta'U - n'(n'-1)\Theta V.$$

We add one or two more examples to illustrate the use which may be made of the covariant F in working questions on conics.

Ex. 1. To find the envelope of the base of a triangle inscribed in a conic V, and two of whose sides touch another conic U.

Let x, y, z be the three sides of such a triangle, then V admits of being transformed into the shape $2(xy + yz + zx)$, and U into the form

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy + 2Bxy,$$

where y and z are the lines which touch U. It is obvious now that $U - BV$ is a conic which will always be touched by the third side z, and we shall show by the invariants that this is a *fixed* conic. We have

$$\Delta' = 2,\ \Theta' = -(a+b+c)^2 + 2B,\ \Theta = 2c(a+b+c)(2ab - B),\ \Delta = -c^2(2ab-B)^2$$

Hence $\Theta^2 - 4\Theta'\Delta = -4B\Delta\Delta'$, and, therefore, the conic $U - BV$ may be written in the form $4\Delta\Delta'U + (\Theta^2 - 4\Theta'\Delta)V$, and is therefore a fixed conic always touched by the third side of the triangle. This immediately gives the condition that the third side of the triangle should also touch U.

Ex. 2. To find the locus of the vertex of a triangle whose three sides touch U, and two of whose vertices move on V.

Let $x'y'z'$ be the vertex; then the method we pursue is to form the equation of the pair of tangents to U through this point: then to form the equation of the lines joining the points where this pair of lines meets V; and lastly, to form the condition that one of these lines (which must be the base of the triangle in question) should touch U. Now if P be the polar of $x'y'z'$ with respect to U, the pair of tangents, as is well known, is $UU' - P^2$. In order to find the chords of intersection with V of the pair of tangents, we form the condition that $UU' - P^2 + \lambda V$ may represent a pair of lines. Now the discriminant of this is $\lambda^3\Delta' + \lambda^2 F + \lambda\Delta U'V'$; the chords of intersection then are $UU' - P^2 + \lambda V$, where λ is one of the roots of the quadratic $\lambda^2\Delta' + \lambda F + \Delta U'V' = 0$. Next, to find the condition that one of these chords should touch U. Now, it has been said already, that the condition that two conics S, S' should touch is obtained by forming the discriminant with respect to μ of the discriminant of $\mu S + S'$. Now, the discriminant of $\mu U + (UU' - P^2 + \lambda V)$ is

$$\mu^2\Delta + \mu(2U'\Delta + \lambda\Theta) + \{U'^2\Delta + \lambda(\Theta U + \Delta V) + \lambda^2\Theta'\}.$$

And the discriminant of this with respect to μ, equated to 0, gives

$$\lambda(4\Delta\Theta' - \Theta^2) + 4\Delta^2 V = 0,$$

which value, substituted in the equation which determines λ, gives for the required locus

$$16\Delta^3\Delta' V - 4\Delta(4\Delta\Theta' - \Theta^2)F + U(4\Delta\Theta' - \Theta^2)^2 = 0,$$

which, as it ought to do, reduces to V when $4\Delta\Theta' = \Theta^2$.

The reader will find an investigation by the same method of the locus of the vertex of a polygon whose sides touch U, and all whose vertices but one move on V; and also of the reciprocal problem ("Philosophical Magazine," April, May, 1857).

141. Let us now pass to a system of three quadrics. In the first place it is evident that if we form the discriminant of $\lambda U + \mu V + \nu W$, we shall get an equation of the third degree in λ, μ, ν, the coefficients of the several powers of which are invariants of the system. The coefficient of $\lambda\mu\nu$ is the invariant

written in full, p. 84. The most important covariant of the system, in addition to the several harmonic conics of each pair, is the Jacobian $\overline{123}$. It was proved (Art. 54) that a point common to all three curves is a double point on the Jacobian. Hence, if three conics have two points common, the Jacobian resolves itself into a line and conic. Thus, in the case of three circles, the Jacobian is the line at infinity, together with the circle cutting all three at right angles ("Conics," p. 324).

The three conics have also a contravariant of the third degree, which may be defined geometrically as the condition that the line $x\xi + y\eta + z\zeta$ should be cut in involution by the three conics; and which, symbolically expressed, is $\overline{\xi 12}.\overline{\xi 23}.\overline{\xi 31}$.

142. It has been already mentioned that a pair of conics has three right lines whose poles are the same with regard to each. We can now easily form the equation of the third degree which determines these three right lines. For, referred to these lines, the two conics would take the form

$$x^2 + y^2 + z^2,\ ax^2 + by^2 + cz^2,$$

and the harmonic conic F will also be of the form $a'x^2 + b'y^2 + c'z^2$, and the Jacobian of these three will be evidently proportional to xyz. Hence, the three lines required are the Jacobian of the three conics U, V, F. For the conic F we may substitute, if we please, the polar of U with regard to V, since we have proved that this last is a linear function of U, V, F (Art. 140). If the two conics are transformed to $x^2 + y^2 + z^2$, $ax^2 + by^2 + cz^2$, we can immediately find the values of a, b, c. For the invariants of the system are

$$\Delta = 1,\quad \Theta = a + b + c,\quad \Theta' = ab + bc + ca,\quad \Delta' = abc.$$

Hence, a, b, c are the roots of $\Delta\lambda^3 - \Theta\lambda^2 + \Theta'\lambda - \Delta' = 0$.

This is the same problem as the transformation of a surface of the second degree to its principal planes, for since $x^2 + y^2 + z^2$ is unaltered by transformation from one set of rectangular axes to another, then if $(a, b, c, d, e, f \between x, y, z)^2$ be the highest terms in the equation of the surface, the problem is to transform simultaneously the two ternary quantics

$$x^2 + y^2 + z^2,\quad (a, b, c, d, e, f \between x, y, z)^2$$

(the latter of which we shall call U) to the form

$$x^2 + y^2 + z^2, \quad Ax^2 + By^2 + Cz^2.$$

The three principal planes are then found by forming the Jacobian of these two quantics, and of their harmonic F, while the new A, B, C are given by the equation in terms of the invariants just written. Or we may substitute for F the polar reciprocal of U with regard to $x^2 + y^2 + z^2$, that is to say

$$(bc - d^2, ca - e^2, ab - f^2, ef - ad, fd - be, de - cf \between x, y, z)^2,$$

which we shall call V, and the three principal planes are then given by the determinant

$$\begin{vmatrix} x, & y, & z \\ \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ \frac{dV}{dx}, & \frac{dV}{dy}, & \frac{dV}{dz} \end{vmatrix}$$

143. We come next to the cubic whose canonical form is $x^3 + y^3 + z^3 + 6mxyz$. Its Hessian $\overline{123}^2$, calculated according to the rules already explained, is found to be

$$H = m^2(x^3 + y^3 + z^3) - (1 + 2m^3)xyz.$$

This is the only independent covariant of the third order; for all other such covariants will be found to be of the form $(x^3 + y^3 + z^3) + 6m'xyz$, and are therefore expressible in the form $U + \lambda H$. Thus, for instance, the Hessian of the Hessian can be expressed in the form just written.

The Hessian of a curve is the locus of a point whose polar conics break up into two right lines, and in the case of a cubic the intersection of these right lines is also a point on the Hessian ("Higher Plane Curves," p. 154). The cubic has also a mixed concomitant $\overline{a12}^2$ (see p. 85), which, for the canonical form, is

$$(yz - m^2x^2)\xi^2 + (zx - m^2y^2)\eta^2 + (xy - m^2z^2)\zeta^2 + 2(m^2yz - mx^2)\eta\zeta$$
$$+ 2(m^2zx - my^2)\zeta\xi + 2(m^2xy - mz^2)\xi\eta.$$

The fundamental invariants of a cubic were discovered by

M. Aronhold, who showed, by a method in substance identical with what we have here called Mr. Cayley's hyperdeterminant method, that they have an invariant of the fourth order, which we shall call S, viz., $\overline{123}.\overline{124}.\overline{234}.\overline{314}$. In fact, this symbol is a symmetrical function of 1, 2, 3, 4, which does not change sign when any of these two are interchanged. Each of these figures enters three times into the symbol, and, therefore, when applied to a cubic it yields an invariant. Its actual value is obtained by multiplying out the symbols for which $\overline{123}$ &c. are abbreviations, and writing the coefficients of the equation instead of the third differentials. The result is given at full length ("Higher Plane Curves," p. 184,* and Mr. Cayley's "Third Memoir on Quantics"). For the canonical form it reduces to $m - m^4$. If the cubic had been written $ax^3 + by^3 + cz^3 + 6mxyz$, S would have been $abcm - m^4$.

144. Having once proved the existence of an invariant, we immediately see that there exists also a contravariant, namely, the evectant of this invariant. For the canonical form this evectant $\xi^3 \dfrac{dS}{da}$ + &c., reduces to $m(\xi^3 + \eta^3 + \zeta^3) + (1 - 4m^3)\xi\eta\zeta$. Mr. Cayley has given three different geometrical interpretations of this function. In the first place, equated to 0, it expresses the condition that the line $x\xi + y\eta + z\zeta$ should be one of the lines which join any point on the Hessian to the *corresponding* point, that is to say, to the intersection of the two lines into which the polar curve of any point on the Hessian breaks up. In other words, the evectant represents the tangential equation of the curve enveloped by the line joining any point on the Hessian to the corresponding point. Secondly, it also expresses the condition that $x\xi + y\eta + z\zeta$ should be one of the lines into which the polar conic of any point on the Hessian breaks up; or, in other words, these lines envelope the very same curve as that enveloped by the line joining two corresponding points. Thirdly, the same equation expresses the condition that $x\xi + y\eta + z\zeta$ should be cut

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* Where, however, the last two groups of terms are printed with wrong signs.

in involution by the system of conics which are polar conics of any point whatever with respect to the cubic. For the proof and development of these theorems, the reader is referred to Mr. Cayley's "Memoir on Curves of the Third Order" ("Philosophical Transactions," 1857, p. 415). If we attempt to form the second evectant of S, $(\xi^3 \frac{d}{da} + \&c.)^2 S$, it will be found to vanish identically.

145. If in the contravariant found in the last Article we substitute $\frac{d}{dx}$ &c. for ξ &c., we get an invariantive symbol of operation. If we apply this to the cubic itself, we fall back on the invariant S, but if we apply it to the Hessian we get an invariant of the sixth order in the coefficients, since the Hessian and the operating symbol each contain the coefficients in the third degree. Aronhold has called this invariant T. T is given at length ("Higher Plane Curves," p. 186), and for the canonical form is $1 - 20m^3 - 8m^6$. Mr. Sylvester has proved (see references on p. 101) that every invariant of a cubic is a rational function of S and T. The knowledge of this enables us at once to see the form of many invariants and covariants. Thus, the discriminant being an invariant of the twelfth order, when expressed in terms of S and T, can only be of the form $S^3 + \lambda T^2$. The actual value, first given by Aronhold, is $T^2 - 64S^3$, as may be proved readily by the canonical form. So again, the Hessian of the Hessian is, as we have already said, of the form $\lambda U + \mu H$, but since this second Hessian is of the ninth order in the coefficients, it follows that λ must be of the eighth, and μ of the sixth. μ, therefore, can only differ by a numerical factor from T, and λ from S^2. The actual value is $4S^2U - TH$. Once more, if we take the mixed concomitant $\overline{a12}^2$, mentioned Art. 143, and substituting $\frac{d}{dx}$ &c. for ξ &c., operate on H, we get a covariant

$$\Phi = \left\{\frac{d^2U}{dy^2}\frac{d^2U}{dz^2} - \left(\frac{d^2U}{dydz}\right)^2\right\}\frac{d^2H}{dx^2} + \&c.$$

This will be evidently of the third degree in the variables, and must, therefore, be of the form $\lambda U + \mu H$. It is also of the

fifth degree in the coefficients; hence λ must be of the fourth, and μ of the second. But there is no invariant of the second order; hence μ must $= 0$, and we can foresee without any calculation that the covariant in question must be identical with SU.

146. The invariant T of course gives rise to a contravariant of the third order, namely, its evectant $\xi^3 \dfrac{dT}{da} + \&c.$, which for the canonical form is $(1 - 10m^3)(\xi^3 + \eta^3 + \zeta^3) - 6(5m^2 + 4m^5)\xi\eta\zeta$. I do not know of any simple geometrical interpretation for this contravariant. T gives rise also to a second contravariant, namely, the second evectant $(\xi^3 \dfrac{d}{da} + \&c.)^2 T$, which will clearly be of the sixth degree in ξ, η, ζ, and of the fourth degree in the coefficients. This is exactly the degree of the polar reciprocal, and, accordingly, Mr. Cayley has verified that the two are identical by forming the evectant of the general value of T, given in my "Higher Plane Curves," and comparing the result with the polar reciprocal. All other contravariants may (with a limitation presently to be noticed) be expressed in terms of the two first evectants of S and T, and of this second evectant of T.

147. We have already said that the only two covariants of the third order are the cubic itself, U, and the Hessian, H. When the quantic is in the canonical form, we can express $x^3 + y^3 + z^3$ and xyz in terms of U and H, and can, of course, express in terms of these two any covariant which is a function only of $x^3 + y^3 + z^3$ and xyz. But as a symmetric function of three quantities cannot be generally expressed unless three things are given, such as their sum, continued product, and product in pairs, it can readily be conceived that to express a covariant in general, we require at least a third fundamental covariant which may afford an expression for $x^3y^3 + y^3z^3 + z^3x^3$. Mr. Cayley has selected as the covariant next in simplicity to the Hessian that obtained by substituting in the bordered Hessian $\overline{a12}^2$, $\dfrac{dH}{dx}, \dfrac{dH}{dy}, \dfrac{dH}{dz}$ for ξ, η, ζ, that is to say,

$$\Theta = \left\{\frac{d^2U}{dy^2}\frac{d^2U}{dz^2} - \left(\frac{d^2U}{dydz}\right)^2\right\}\left(\frac{dH}{dx}\right)^2 + \&c.$$

This is easily seen to be of the sixth degree in the variables, and of the eighth in the coefficients, and for the canonical form is

$$(1+8m^3)^2(y^3z^3+z^3x^3+x^3y^3)-(2m^3+m^6)U^2+(2m-5m^4)UH-3m^2H^2.$$

Every covariant can, in general, be expressed in terms of U, H, Θ. Thus, the product of the nine tangents at the points of inflexion, expressed by the help of the canonical form in terms of these quantities, is found to be

$$\{U-(1+8m^3)x^3\}\{U-(1+8m^3)y^3\}\{U-(1+8m^3)z^3\}$$
$$= H^3 - 8SU^2H + \Theta U.$$

But a covariant need not necessarily be a *rational* function of U, H, Θ. Thus, every perfectly symmetric function of the roots of an equation can be expressed as a rational function of its coefficients, but not so one which is symmetric with the exception of its sign; for example, the simple product of the differences of its roots. We see then that there exist covariants analogous to Hermite's invariant (Art. 136) not expressible as rational functions of U, H, Θ. For example, the nine harmonic polars of the points of inflexion have for their equation, in the canonical form, $(x^3-y^3)(y^3-z^3)(z^3-x^3)$, which cannot be expressed in terms of U, H, Θ, although its square evidently can. Similar remarks apply to contravariants.

There is another skew mixed concomitant $\overline{a12}^2\overline{a13}$ of the third degree in both variables, and in the coefficients, of which, however, no important use has as yet been made. For the canonical form it is

$$\begin{aligned}&\xi^3(y^3-z^3)+\eta^3(z^3-x^3)+\zeta^3(x^3-y^3)-\xi^2\eta\{(1+8m^3)x^2y\\&\quad+6my^2z+12m^2z^2x\}+\xi\eta^2\{(1+8m^3)xy^2+6mx^2z+12m^2z^2y\}\\&\quad-\eta^2\zeta\{(1+8m^3)y^2z+6mz^2x+12m^2x^2y\}+\eta\zeta^2\{(1+8m^3)yz^2\\&\quad+6my^2x+12m^2x^2z\}+\zeta\xi^2\{(1+8m^3)zx^2+6mz^2y+12m^2y^2x\}\\&\quad-\zeta^2\xi\{(1+8m^3)z^2x+6mx^2y+12m^2y^2z\}.\end{aligned}$$

148. We conclude the subject of cubics by describing a method by which Mr. Sylvester has obtained S and T simultane-

ously. It will first, however, be necessary to say something as to a class of invariants which we have not yet mentioned. Let there be a function homogeneous in x, y, z, and also homogeneous in another set of variables α, β, γ. Let $\overline{123}$ have the same meaning as before, and let $\overline{123}'$ denote the same thing with respect to the differentials with regard to α, β, γ; then it is evident that if we operate on the function with any power of the product $\overline{123}.\overline{123}'$, we shall get a covariant or invariant. Thus, for example, take the function

$$ax\xi + by\eta + cz\zeta + dy\zeta + d'z\eta + ez\xi + e'x\zeta + fx\eta + f'y\xi,$$

and, operating with $\overline{123}.\overline{123}'$, we get an invariant

$$abc + def + d'e'f' - add' - bee' - cff'.$$

When the two sets of variables are made identical, and the accents omitted, this becomes the discriminant of a quadric. It will be useful if we work out in like manner the result of the operation $\overline{123}^2.\overline{123}'^2$ applied to a function of the second degree both in x, y, z, and in α, β, γ. I write for brevity x instead of $\dfrac{d}{dx}$ &c.; then, multiplying out the square of the product of the two determinants $(xy'z'')(\alpha\beta'\gamma'')$, and afterwards omitting the accents, the result is found to be

$$\begin{aligned}
&x^2\alpha^2.y^2\beta^2.z^2\gamma^2 + (x^2\alpha^2.y^2\gamma^2.z^2\beta^2 + y^2\beta^2.x^2\gamma^2.z^2\alpha^2 + z^2\gamma^2.x^2\beta^2.y^2\alpha^2) \\
&\quad + 2\,(x^2\alpha^2.yz\beta\gamma.yz\beta\gamma + y^2\beta^2.zx\gamma\alpha.zx\gamma\alpha + z^2\gamma^2.xy\alpha\beta.xy\alpha\beta) \\
&\quad - 2\,(x^2\alpha^2.y^2\beta\gamma.z^2\beta\gamma + x^2\alpha^2.yz\beta^2.yz\gamma^2 + y^2\beta^2.z^2\alpha\gamma.x^2\alpha\gamma + y^2\beta^2. \\
&\quad xz\gamma^2.xz\alpha^2 + z^2\gamma^2.x^2\alpha\beta.y^2\alpha\beta + z^2\gamma^2.xy\alpha^2.xy\beta^2) + 2\,(x^2\alpha\beta.y^2\beta\gamma. \\
&\quad z^2\gamma\alpha + x^2\alpha\gamma.y^2\alpha\beta.z^2\beta\gamma + \alpha^2xy.\beta^2yz.\gamma^2zx + \alpha^2xz.\beta^2xy.\gamma^2yz) \\
&\quad + 2\,(x^2\beta\gamma.y^2\alpha\beta.z^2\alpha\gamma + y^2\alpha\gamma.z^2\beta\gamma.x^2\beta\alpha + z^2\alpha\beta.x^2\alpha\gamma.y^2\beta\gamma \\
&\quad + \alpha^2yz.\beta^2xy.\gamma^2xz + \beta^2xz.\gamma^2yz.\alpha^2yx + \gamma^2xy.\alpha^2xz.\beta^2yz) + \\
&\quad 4\,(x^2\alpha\beta.yz\alpha\beta.yz\gamma^2 + x^2\alpha\gamma.yz\alpha\gamma.yz\beta^2 + y^2\alpha\beta.zx\alpha\beta.zx\gamma^2 + \\
&\quad y^2\beta\gamma.zx\beta\gamma.zx\alpha^2 + z^2\gamma\alpha.xy\gamma\alpha.xy\beta^2 + z^2\gamma\beta.xy\gamma\beta.xy\alpha^2) - \\
&\quad 2\,(x^2\beta^2.y^2\gamma\alpha.z^2\gamma\alpha + x^2\beta^2.yz\gamma^2.yz\alpha^2 + x^2\gamma^2.y^2\alpha\beta.z^2\alpha\beta + x^2\gamma^2. \\
&\quad yz\alpha^2.yz\beta^2 + y^2\alpha^2.z^2\beta\gamma.x^2\beta\gamma + y^2\alpha^2.zx\beta^2.zx\gamma^2 + y^2\gamma^2.zx\alpha^2.zx\beta^2 \\
&\quad + y^2\gamma^2.x^2\alpha\beta.z^2\alpha\beta + z^2\alpha^2.xy\beta^2.xy\gamma^2 + z^2\alpha^2.x^2\beta\gamma.y^2\beta\gamma + z^2\beta^2. \\
&\quad xy\alpha^2.xy\gamma^2 + z^2\beta^2.x^2\alpha\gamma.y^2\alpha\gamma) - 4\,(x^2\alpha\beta.yz\alpha\gamma.yz\beta\gamma + x^2\alpha\gamma. \\
&\quad yz\beta\alpha.yz\beta\gamma + y^2\beta\alpha.zx\gamma\alpha.zx\gamma\beta + y^2\beta\gamma.zx\alpha\beta.zx\alpha\gamma + z^2\alpha\gamma. \\
&\quad xy\beta\alpha.xy\beta\gamma + z^2\beta\gamma.xy\alpha\beta.xy\alpha\gamma + \alpha^2xy.\beta\gamma zx.\beta\gamma zy + \alpha^2xz. \\
&\quad \beta\gamma yx.\beta\gamma yz + \beta^2yx.\gamma\alpha zx.\gamma\alpha zy + \beta^2yz.\gamma\alpha xy.\gamma\alpha xz + \gamma^2xz.
\end{aligned}$$

$\alpha\beta yx.\alpha\beta yz + \gamma^2 yz.\alpha\beta xy.\alpha\beta xz) + (x^2\beta^2.y^2\gamma^2.z^2\alpha^2 + x^2\gamma^2.y^2\alpha^2.z^2\beta^2) + 4xy\alpha\beta.yz\beta\gamma.zx\gamma\alpha + 2(x^2\beta^2.yz\gamma\alpha.yz\gamma\alpha + x^2\gamma^2.yz\alpha\beta.yz\alpha\beta + y^2\alpha^2.zx\beta\gamma.zx\beta\gamma + y^2\gamma^2.zx\alpha\beta.zx\alpha\beta + z^2\alpha^2.xy\beta\gamma.xy\beta\gamma + z^2\beta^2.xy\alpha\gamma.xy\alpha\gamma) + 4(xy\alpha\beta.yz\alpha\gamma.xz\beta\gamma + yz\beta\gamma.zx\alpha\beta.xy\alpha\gamma + zx\gamma\alpha.xy\beta\gamma.yz\beta\alpha) + 4(xy\alpha\beta.xy\gamma^2.z^2\alpha\beta + yz\beta\gamma.yz\alpha^2.x^2\beta\gamma + zx\alpha\gamma.zx\beta^2.y^2\alpha\gamma) + 2(x^2\beta\gamma.y^2\alpha\gamma.z^2\alpha\beta + \alpha^2 yz.\beta^2 zx.\gamma^2 xy) + 4(xz\alpha\beta.xy\beta\gamma.yz\gamma\alpha + xy\alpha\gamma.xz\beta\gamma.yz\alpha\beta) - 4(xy\gamma^2.xz\alpha\beta.yz\alpha\beta + xz\beta^2.xy\alpha\gamma.yz\alpha\gamma + yz\alpha^2.zx\beta\gamma.xy\beta\gamma + x^2\beta\gamma.yz\gamma\alpha.yz\alpha\beta + y^2\gamma\alpha.zx\alpha\beta.zx\beta\gamma + z^2\alpha\beta.xy\beta\gamma.xy\gamma\alpha)$.

149. To apply the above principles to ternary cubics we have only to recollect that they have a mixed concomitant $\overline{a12}^2$, which is a homogeneous function of the second degree in both sets of variables; to which we may add $\lambda(x\xi + y\eta + z\zeta)^2$, which is a function of the same kind. The invariant, then, of the sum, calculated as in the last Article, will be an invariant of the cubic, and the coefficients of the several powers of λ must be separately invariants. Thus, for the canonical form, the mixed concomitant in question is

$$(\lambda - m^2)(x^2\xi^2 + y^2\eta^2 + z^2\zeta^2) + 2(\lambda + m^2)(yz\eta\zeta + zx\zeta\xi + xy\xi\eta) + (yz\xi^2 + zx\eta^2 + xy\zeta^2) - 2m(x^2\eta\zeta + y^2\zeta\xi + z^2\xi\eta).$$

We give in order the result of operating on the above with the several groups of terms given in the last Article.

$$8(\lambda - m^2)^3,\ 0,\ 12(\lambda - m^2)(\lambda + m^2)^2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 4(\lambda + m^2)^3,\ 0,\ 0,\ -24m(\lambda + m^2),\ -16m^3,\ 0,\ 2,$$

which terms, expanded and arranged, are

$$12\lambda^3 - 12(m - m^4)\lambda + 1 - 20m - 8m^6,$$

or

$$12\lambda^3 - 12S\lambda + T.$$

150. We return now to the case of three quadrics, U, V, W, in order to give a discussion of their eliminant due to Mr. Sylvester, and to examine whether it can be expressed in terms of simpler invariants. Eliminants belong to a class of invariants which he calls *combinants*, which are unaltered not merely when the variables undergo linear transformation, but also when for the quantics themselves we substitute any linear functions of them. Thus, the eliminant of

$$\lambda U + \mu V + \nu W,\ \lambda' U + \mu' V + \nu' W,\ \lambda'' U + \mu'' V + \nu'' W,$$

manifestly differs only by a constant from the eliminant of U, V, W.* Now, it has been already remarked that the coefficients of the several powers of λ, μ, ν in the discriminant of $\lambda U + \mu V + \nu W$ are invariants; but if we form the invariants of this discriminant considered as a cubic function of λ, μ, ν, these must not only be invariants, but combinants of the system U, V, W, since they are unaltered when λ, μ, ν receive linear transformations. The invariant† S of this cubic is easily seen to be of the fourth degree in the coefficients of each of the quantics; it is therefore a combinant of precisely the degree of the eliminant, and the question naturally presents itself, are the two identical? The enormous length of the eliminant in general would render the comparison next to impossible, were it not for the advantage gained by the fact that the eliminant is a combinant.

151. We may, without loss of generality, suppose two of the quantics reduced to their canonical form

$$x^2 + y^2 + z^2,\ ax^2 + by^2 + cz^2,$$

while the third remains in the general form. But again, we may substitute for the first two any linear functions of them; and, therefore, may substitute for the first two, the result of eliminating z and y alternately between them, so that the first pair of quantics can be brought to the form $x^2 - z^2$, $y^2 - z^2$, while the third may, by the help of these two, be reduced to the form $z^2 + 2lyz + 2mzx + 2nxy$. And any results deducible for this simple system will be true in general. But the elimi-

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* Hence, also, in addition to the ordinary differential equations of invariants, combinants satisfy others. For if a, b, c, &c., be the coefficients of U; a', b', c', &c., the corresponding coefficients of V; we must plainly have $a'\dfrac{dI}{da} + b'\dfrac{dI}{db} + c'\dfrac{dI}{dc} + \text{\&c.} = 0$. A combinant must vanish if any two of the quantics become identical; and it will be a function of the determinants $(ab'c'')$ &c. formed with every three sets of corresponding coefficients of the three quantics.

† We get the very same quantity if we form the S of the Jacobian of the system of quadrics.

nant of the system got by solving for x, y, z from the first pair, and substituting in the third quantic, is easily seen to be

$$(1 + 2l + 2m + 2n)(1 - 2l + 2m - 2n)(1 + 2l - 2m - 2n)(1 - 2l - 2m + 2n) = 1 - 8(l^2 + m^2 + n^2) + 16(l^4 + m^4 + n^4 - 2l^2m^2 - 2m^2n^2 - 2n^2l^2) + 64lmn.$$

Next, to form the discriminant of $\lambda U + \mu V + \nu W$, which is found to be

$$\nu^3(2lmn - n^2) + (n^2 - l^2)\nu^2\lambda + (n^2 - m^2)\nu^2\mu - \lambda^2\mu - \lambda\mu^2 + \lambda\mu\nu.$$

This is a ternary cubic with regard to λ, μ, ν; and its S, worked out by the formula ("Higher Plane Curves," p. 184) will be found to be

$$1 - 8(l^2 + m^2 + n^2) + 16(l^4 + m^4 + n^4 - l^2m^2 - m^2n^2 - n^2l^2) + 48lmn.$$

It appears, then, that this latter invariant is *not* identical with the eliminant. If, however, we eliminate the term lmn, calling the eliminant of the system of quadrics E, we have

$$4S - 3E = 1 - 8(l^2 + m^2 + n^2) + 16(l^4 + m^4 + n^4 + 2l^2m^2 + 2m^2n^2 + 2n^2l^2) = \{1 - 4(l^2 + m^2 + n^2)\}^2.$$

Thus, we perceive that *the eliminant differs from a multiple of the invariant S only by a term which is the square of an invariant of half the dimensions in the coefficients.*

152. Mr. Sylvester obtains this invariant as follows:—Form the polar reciprocal of $\lambda U + \mu V + \nu W$, which will evidently be of the second degree both in the variables ξ, η, ζ, and also in λ, μ, ν. Then the invariant of this function, as in Art. 148, will give the combinant required, which will involve the coefficients of each of the three quantics in the second degree. I had actually worked out by the formula of Art. 148 the general expression of this combinant, when another and much simpler method of doing the same thing suggested itself. We know that the three quadrics have a cubic covariant, viz., the Jacobian, and we know also (Art. 141) that they have a cubic contravariant $\xi\overline{12}.\xi\overline{23}.\xi\overline{31}$. We have then only to substitute in the contravariant differential symbols for ξ, η, ζ, and operate on the Jacobian, when we obtain the combinant required. Since the Jacobian and the contravariant contain each of them in the first

degree the coefficients of each of the three quantics, the combinant will contain these coefficients in the second degree. It may be remarked, in passing, that the Jacobian is the locus of the intersection of any pair of right lines which can be represented by $\lambda U + \mu V + \nu W$, whilst the contravariant in question expresses the condition that $x\xi + y\eta + z\zeta$ should be one of those right lines. Now the general expression for the Jacobian is (using $(ab'c'')$ &c. to denote determinants, as at p. 3),

$$(ae'f'')x^3 + (bf'd'')y^3 + (cd'e'')z^3 - \{(ab'e'') + (af'd'')\}x^2y + \{(ac'f'') + (ae'd'')\}x^2z + \{(ba'd'') + (bf'e'')\}y^2x - \{(bc'f'') + (bd'e'')\}y^2z - \{(ca'd'') + (ce'f'')\}z^2x + \{(cb'e'') + (cd'f'')\}z^2y - \{(ab'c'') + 2(de'f'')\}xyz,$$

while the contravariant is

$$(bc'd'')\xi^3 + (ca'e'')\eta^3 + (ab'f'')\zeta^3 - \{(bc'e'') + 2(cd'f'')\}\xi^2\eta + \{(cb'f'') + 2(bd'e'')\}\xi^2\zeta + \{(ac'd'') + 2(ce'f'')\}\eta^2\xi - \{(ca'f'') + 2(ae'd'')\}\eta^2\zeta - \{(ab'd'') + 2(bf'e'')\}\zeta^2\xi + \{(ba'e'') + 2(af'd'')\}\zeta^2\eta + \{(ab'c'') - 4(de'f'')\}\xi\eta\zeta.$$

From these two the combinant is found to be*

$$(ab'c'')^2 + 4(ab'd'')(ac'd'') + 4(bc'e'')(ba'e'') + 4(ca'f'')(cb'f'') + 8(ad'e'')(bd'e'') + 8(ad'f'')(cd'f'') + 8(ce'f'')(be'f'') - 8(ae'f'')(bc'd'') - 8(bf'd'')(ca'e'') - 8(ab'f'')(cd'e'') + 4(ab'c'')(de'f'') - 8(de'f'')^2.$$

In the particular case considered in Art. 151, all the terms vanish except the first four, and the combinant reduces, as we expected, to $1 - 4(l^2 + m^2 + n^2)$.

153. We conclude this Lesson by a brief account of quartics, the theory of which has been very imperfectly studied. The order of every invariant must be of the form $3m$. That of the third order has been already given, p. 85. That of the sixth order can easily be written down in the form of a determinant by taking the six second differentials and eliminating dialytically between them the six quantities x^2, y^2, &c. Assuming that the equation of the polar reciprocal is an evectant of the

* N. B.—We have identically
$$(ad'f'')(bc'e'') + (ad'e'')(cb'f'') + (ab'c'')(de'f'') = (ae'f'')(bc'd'').$$

form $(\xi^4 \frac{d}{da} + \&c.)^3 I$, I will be an invariant of the ninth order, which may be worked out by the help of the known equation of the polar reciprocal. I know nothing as to the other invariants of the quartic, except that the discriminant is of the twenty-seventh order, but this can most probably be expressed in terms of simple invariants.

We have already (p. 85, and "Higher Plane Curves," p. 101) two contravariants of the quartic of the degrees respectively 4 and 6 in the variables; and of the orders 2 and 3 in the coefficients. The polar reciprocal can be expressed in terms of these two. Of course, we can form as many more contravariant quartics as we have independent invariants, by taking the evectant of each. As covariants, we have the Hessian of the degree six, and order three; while, by taking the S and T of the first emanant, which is of course a cubic, we get a covariant quartic of the fourth order in the coefficients, and a sextic of the sixth order. We get a number of other covariant quartics either by forming the contravariant of any of the contravariant quartics already noticed, or by forming a covariant S to $U + \lambda S$: and in like manner, from any one covariant sextic we can derive a number of others by forming, according to the same rule, the covariant of $U + \lambda S$ where S may be any of the covariant quartics. Without knowing the number of independent invariants, it seems impossible to say how many of these covariant quartics or sextics are independent. It is easily proved that the covariant quartic got by forming the contravariant of the simplest contravariant quartic is equal to the sum of the covariant which we have called S, together with the product of U, by a multiple of the simplest invariant.

Again, if in the contravariant quartic which is of the second order, we substitute differentials for the variables, and operate on the Hessian, we get a covariant conic of the fifth order in the coefficients. While again, if we treat the quartic itself in like manner, and operate on the simplest sextic contravariant, we get a contravariant conic of the fourth order in the coefficients. So again, there will be another covariant conic of the

eighth order, and a contravariant of the seventh, &c. Some other covariants are of importance in the theory of quartics. Thus, if we form for the quartic the covariant which we have called Θ (Art. 147), it may easily be seen to be of the fourteenth degree. If we form the covariant called Φ (Art. 145), it may be seen to be of the eighth degree. The curve which determines the points of contact of double tangents is $\Theta = 3H\Phi$ ("Higher Plane Curves," p. 89).

LESSON XV.

THEORY OF QUADRICS IN GENERAL.

154. In the last two Lessons we have given the principal applications of the preceding theory to binary and ternary quantics. With regard to quantics in any greater number of variables, we proceed no further than the case where the quantic is of the second degree, of which we shall give some account in this Lesson. The theory of quadrics is nearly the same, no matter what the number of variables; we shall, therefore, have to repeat something of what has been already said (Art. 138, &c.), but we shall enter into a little fuller development.

Perhaps the most easily intelligible notation for the coefficients of a quadric in general is to use a double suffix, and to write the coefficients of x^2, y^2, &c., $a_{1,1}$, $a_{2,2}$, $a_{3,3}$, &c., while the coefficients of xy, xz, yz, &c. are $2a_{1,2}$, $2a_{1,3}$, $2a_{2,3}$, &c.; $a_{1,2}$ and $a_{2,1}$, being identical in this notation. Then it is evident that the discriminant of the quadric is the *symmetrical* determinant (see p. 7),

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3} & \&c. \\ a_{2,1}, & a_{2,2}, & a_{2,3} & \&c. \\ a_{3,1}, & a_{3,2}, & a_{3,3} & \&c. \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

for which we may use the abbreviation $\Sigma a_{1,1}a_{2,2}a_{3,3}$ &c. The above is the only invariant of the quantic. It has, however, a

contravariant, namely, the evectant of the discriminant, which (as in Art. 138) may be written in the form of a determinant by bordering the above matrix horizontally, and vertically with the contragredient variables. Thus, in the case of a quaternary quadric the evectant $\left(\alpha^2 \frac{d}{da_{1,1}} + \alpha\beta \frac{d}{da_{1,2}} + \&c.\right)I$ may be written

$$\begin{vmatrix} 0, & \alpha, & \beta, & \gamma, & \delta \\ \alpha, & a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4} \\ \beta, & a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4} \\ \gamma, & a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4} \\ \delta, & a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4} \end{vmatrix}$$

or rather, this determinant is equal to the evectant with its sign changed. For it is easy to see by the rule of signs that the coefficient of α^2 in the determinant last written is – the determinant $\Sigma a_{2,2} a_{3,3} a_{4,4}$: but this determinant is the differential of the discriminant with respect to $a_{1,1}$.

When the discriminant vanishes, the evectant becomes a perfect square; for the proof given for the case of a ternary quantic, p. 68, applies word for word in general, and shows that the evectant of the discriminant of a quantic of the n^{th} degree in any number of variables becomes a perfect n^{th} power, in case the discriminant vanishes. It is proved, then, that when the discriminant vanishes, the determinant last written becomes a perfect square, and the sign will of course be the same as that of the factor which multiplies α^2, that is to say, it will be opposite to the sign of the determinant $\Sigma a_{2,2} a_{3,3} a_{4,4}$.

155. The property just obtained is worth stating as a general property of a symmetrical determinant. The evectant determinant written in the last Article may be taken as the general form of a symmetrical determinant whose first term vanishes; the discriminant is the first minor obtained from this by erasing the outside line and column, while the determinant $\Sigma a_{2,2} a_{3,3} a_{4,4}$ is the second minor obtained by erasing the outside line and column of the discriminant. What we have proved, then, is that when the first minor (as just explained) vanishes of a symmetrical determinant wanting the first term, then the

determinant and the second minor will have opposite signs. But it is easy to see that this must be equally true if the first term of the given symmetrical determinant does *not* vanish. For in the expansion of the determinant, the first term is multiplied (see Art. 22) by the first minor which vanishes by hypothesis, and therefore, the presence or absence of the first term will not affect the result.

156. We have already (p. 111) spoken of the problem of transforming the equation of a surface of the second degree to its principal planes. We have seen that since the quantity $x^2 + y^2 + z^2$ remains unchanged by transformation from one rectangular system to another, the problem becomes, to transform simultaneously $(a, b, c, d, e, f)(x, y, z)^2$ and $x^2 + y^2 + z^2$, so that the latter may still be of the form $x^2 + y^2 + z^2$, and that the former may become $Ax^2 + By^2 + Cz^2$. In like manner, a quadratic function of any number of variables may be transformed in an infinity of ways to a sum of squares (Art. 118), but what we may call the *orthogonal* transformation is to transform simultaneously a given quadratic function, and $x^2 + y^2 + z^2 + w^2 + \&c.$, so that the latter remaining of the same form, the former may become $Ax^2 + By^2 + Cz^2 + Dw^2 + \&c.$

The transformation is effected in the same manner as for ternary quadrics (Art. 142). Let the given quadric be U, and let $x^2 + y^2 + z^2 + \&c. = V$, then if we form the discriminant of $U + \lambda V$, we shall have a function in which all the coefficients of λ will be invariants, and if we put this discriminant $= 0$, and solve for λ, we must get the same roots, no matter how the quantic is transformed. The discriminant of $U + \lambda V$ is at once written down by adding λ to the coefficients of x^2, y^2, z^2 in the determinant of Art. 154, when we get

$$\begin{vmatrix} a_{1,1}+\lambda, & a_{1,2}, & a_{1,3}, & a_{1,4} & \&c. \\ a_{2,1}, & a_{2,2}+\lambda, & a_{2,3}, & a_{2,4} & \&c. \\ a_{3,1}, & a_{3,2} & a_{3,3}+\lambda, & a_{3,4} & \&c. \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}+\lambda & \&c. \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} *$$

* The equation got by equating this determinant to 0, is of considerable importance in analysis. It occurs in the determination of the secular inequalities of the planets (see Laplace, "Mécanique Celeste," Part I., Book ii., Art. 56).

But now in the form to which we tranform the quantic, since every coefficient of the form $a_{1,2}$ vanishes, the determinant reduces to the single product $(A+\lambda)(B+\lambda)(C+\lambda)$ &c. Since, then, this expression $(A+\lambda)(B+\lambda)$ &c. is identical with the determinant last written, it follows that the new coefficients A, B, C, &c., are given immediately as the roots with the signs changed of the equation in λ, got by putting that determinant $= 0$. The roots of this equation will always be real; and we can at once determine by Descartes's rule how many of them are positive, and how many negative, and so ascertain (what has been proved, Art. 48, to be a fixed number) the number of positive and negative squares into which the function is transformable.

157. Several proofs have been given of the fact just stated, namely, that the roots of the equation of the last Article are all real: the following is, perhaps, as simple as any. Take the determinant of the last Article, and form from it a minor, as in Art. 155, by erasing the outside line and column: form from this again another minor by the same rule, and so on. We shall thus have a series of functions of λ, whose degrees regularly diminish from the n^{th} to the 1^{st}; and we may take any positive constant to complete the series. Now, if we substitute successively in this series any two values of λ, and count in each case the variations of sign as in Sturm's theorem, it is easy to see that the difference in the number of variations cannot exceed the number of roots of the equation of the n^{th} degree which lie between the two assumed values of λ. This appears at once from what was proved in Art. 155, that if λ be taken so as to make any of these minors vanish, the two adjacent functions in the series will have opposite signs. It follows, then, precisely as in the proof of Sturm's theorem, that if we diminish λ regularly from $+\infty$ to $-\infty$, then as λ passes through a root of any of these minors, the number of variations in the series will not be affected; and that a change in the number of variations can only take place when λ passes through a root of the first equation, namely, that in which λ enters in the n^{th} degree. The total number of variations, therefore, cannot exceed the number of real roots of this equation.

But obviously, in all these functions the sign of the highest power of λ is positive; hence, when we substitute $+\infty$, we get no variation; when we substitute $-\infty$, the terms become alternately positive and negative, and we get n variations; the equation we are discussing must, therefore, have n real roots. It is easy to see, in like manner, that the roots of every one of the series of functions are all real, and that the roots of each are interposed as limits between the roots of the function next above it in the series.

158. It will be perceived that in the preceding Article we have substituted, for the functions of Sturm's theorem, another series of functions possessing the same fundamental property: that when one vanishes, the two adjacent have opposite signs. Mr. Borchardt, however, has proved (see "Liouville's Journal," vol. xii., p. 50) that the roots of the equation in Art. 154 are all real, by a direct application of Sturm's theorem; and as the principles involved in his proof are all worth knowing for their own sake, we give his demonstration here.

The first principle which it will be necessary to use is a theorem given by Mr. Sylvester ("Philosophical Magazine," December, 1839), that the several functions in Sturm's series, expressed in terms of the roots of the given equation, differ only by positive square multipliers from the following. The first two (namely, the function itself, and its derived) are of course, $(x-\alpha)(x-\beta)(x-\gamma)$ &c., $\Sigma(x-\beta)(x-\gamma)$ &c.; and the remaining ones are

$$\Sigma(\alpha-\beta)^2(x-\gamma)(x-\delta) \text{ \&c.}; \ \Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(x-\delta) \text{ \&c.},$$

where we take the product of any k factors of the given equation, and, multiplying by the product of the squares of differences of all the roots not contained in these factors, form the corresponding symmetric function. We commence by proving this theorem.*

* I suppose that Mr. Sylvester must have originally divined the form of these functions from the characteristic property of Sturm's functions, viz., that if the equation has two equal roots $\alpha=\beta$, every one of them must become divisible by

159. In the first place, let U be the function, V its first derived, R_2, R_3, &c. the series of Sturm's remainders: then it is easy to see that any one of them can be expressed in the form $AV - BU$. For, from the fundamental equations

$$U = Q_1V - R_2,\ V = Q_2R_2 - R_3,\ R_2 = Q_3R_3 - R_4 \text{ \&c.},$$

we have

$$R_2 = Q_1V - U,$$
$$R_3 = Q_2R_2 - V = (Q_2Q_1 - 1)V - Q_2U,$$
$$R_4 = (Q_3Q_2 - 1)R_2 - Q_3V = (Q_1Q_2Q_3 - Q_1 - Q_3)V - (Q_2Q_3 - 1)U,$$

and so on. We have then in general* $R_k = AV - BU$, where, since all the Q's are of the first degree in x, it is easy to see that A is of the degree $k - 1$, and B of the degree $k - 2$, while R_k is of the degree $n - k$.

But now this property would suffice to determine R_2, R_3, &c. directly. Thus, if in the equation $R_2 = Q_1V - U$ we assume $Q_1 = ax + b$, where a and b are unknown constants, the condition that the coefficients of the two highest powers of x on the right-hand side of the equation must vanish (since R_2 is only of the

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$x - a$. Consequently, if we express any one of these functions as the sum of a number of products $(x - \alpha)(x - \beta)$ &c., every product which does not include either $x - \alpha$ or $x - \beta$ must be divisible by $(\alpha - \beta)^2$; and it is evident in this way that the theorem *ought* to be true. The method of verification here employed does not differ essentially from Mr. Sturm's proof, "Liouville," vol. vii., p. 356. Mr. Sylvester himself deduces his theorem as a particular case of a more general one concerning functions formed in the process of eliminating between any two equations. His valuable memoir ("Philosophical Transactions," 1853) contains much of which I am tempted to give an account here—

"Extremo ni jam sub fine laborum
Vela traham, et terris festinem advertere proram."

* The theory of continued fractions which we are virtually applying here shows that if we have $R_k = A_kV - B_kU$, $R_{k+1} = A_{k+1}V - B_{k+1}U$, then $A_kB_{k+1} - A_{k+1}B_k$ is constant and $= 1$. In fact, since $R_{k+1} = Q_kR_k - R_{k-1}$, we have

$$A_{k+1} = Q_kA_k - A_{k-1},\quad B_{k+1} = Q_kB_k - B_{k-1},$$

whence

$$A_kB_{k+1} - A_{k+1}B_k = A_{k-1}B_k - A_kB_{k-1},$$

and by taking the values in the first two equations above, namely, where $k = 2$, and $k = 3$, we see that the constant value $= 1$.

degree $n-2$) is sufficient to determine a and b. And so in general, if in the function $AV-BU$ we write for A the most general function of the $k-1^{st}$ degree containing k constants, and for B the most general function of the $k-2^{nd}$ degree, containing $k-1$ constants, we appear to have in all $2k-1$ constants at our disposal, and have in reality one less, since one of the coefficients may by division be made $=1$.* We have then just constants enough to be able to make the first $2k-2$ terms of the equation vanish, or to reduce it from the degree $n+k-2$ to the degree $n-k$. The problem, then, to form a function of the degree $n-k$, and expressible in the form $AV-BU$, where A and B are of the degrees $k-1$, $k-2$, is perfectly definite, and admits but of one solution. If, then, we have ascertained that any function R_k is expressible in the form $AV-BU$, where A and B are of the right degree, we can infer that R_k must be identical with the corresponding Sturm's remainder, or at least only differ from it by a constant multiplier. It is in this way that we shall identify with Sturm's remainders the expressions in terms of the roots, Art. 158.

160. Let us now, to fix the ideas, take any one of these functions, suppose

$$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(x-\delta)(x-\epsilon)\ \&c.,$$

and we shall prove that it is of the form $AV-BU$, where in the example chosen A is to be of the second degree, and B of the first in x. Now we can immediately see what we are to assume for the form of A, by making $x=\alpha$ on both sides of the equation. The right-hand side of the equation will then become $A(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon)$ &c. since U vanishes; and the left-hand side will become

$$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\delta)(\alpha-\epsilon)\ \&c.$$

It follows, then, that the supposition $x=\alpha$ must reduce A to the

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* Just as the six constants in the most general equation of a conic are only equivalent to five independent constants, and only enable us to make the curve satisfy five conditions.

form $\Sigma(\beta-\gamma)^2(a-\beta)(a-\gamma)$, and it is at once suggested that we ought to take for A the symmetric function

$$\Sigma(\beta-\gamma)^2(x-\beta)(x-\gamma).$$

And in like manner, in the general case, we are to take for A the symmetric function of the product of $k-1$ factors of the original equation multiplied by the product of the squares of the differences of all the roots which enter into these factors. It will not be necessary to our purpose actually to determine the coefficients in B, which we shall therefore write down in its most general form. Let us then write down

$$\Sigma(a-\beta)^2(\beta-\gamma)^2(\gamma-a)^2(x-\delta)\ \&c. = \Sigma(a-\beta)^2(x-a)(x-\beta) \\ \times\Sigma(x-\beta)(x-\gamma)\ \&c. + (ax+b)(x-a)(x-\beta)\ \&c.,$$

which we are to prove is an identical equation. Now, since an equation of the p^{th} degree can only have p roots, if such an equation is satisfied by more than p values of x, it must be an identical equation, or one in which the coefficients of the several powers of x separately vanish. But the equation we have written down is satisfied for each of the n values $x=a$, $x=\beta$, &c., no matter what the values of a and b may be. And if we substitute any other two values of x, then, by solving for a and b from the equations so obtained, we can determine a and b so that the equation may be satisfied for these two values. It is, therefore, satisfied for $n+2$ values of x, and since it is only an equation of the $n+1^{st}$ degree, it must be an identical equation. And the corresponding equation in general, which is of the $n+k-1$ degree, is satisfied immediately for any of the n values $x=a$ &c.; while B being of the $k-1$ degree we can determine the k constants which occur in its general expression, so that the equation may be satisfied for k other values; the equation is, therefore, an identical equation.

161. We have now proved that the functions written in Art. 158 being of the form $AV-BU$ are either identical with Sturm's remainders, or only differ from them by constant factors. It remains to find out the value of these factors, which is an essential matter, since it is on the signs of the functions that

everything turns. Calling Sturm's remainders, as before, R_2, R_3, &c., let Mr. Sylvester's forms (Art. 158) be T_2, T_3, &c., then we have proved that the latter are of the form $T_2 = \lambda_2 R_2$, $T_3 = \lambda_3 R_3$, &c., and we want to determine λ_2, λ_3, &c. We can at once determine λ_2 by comparing the coefficients of the highest powers of x on both sides of the identity $T_2 = A_2 V - B_2 U$; for x^n does not occur in T_2, while in V the coefficient of x^{n-1} is n, and the coefficient of x is also n in A_2, which $= \Sigma(x - a)$; hence $B_2 = n^2$. But the equation $T_2 = A_2 V - B_2 U$ must be identical with the equation $R_2 = Q_1 V - U$ multiplied by λ_2; we have, therefore, $\lambda_2 = n^2$.

To determine in general λ_k, it is to be observed that since any equation $T_k = A_k V - B_k U$ is λ_k times the corresponding equation for R_k, and since in the latter case it was proved (note, p. 128) that $A_k B_{k+1} - A_{k+1} B_k = 1$, the corresponding quantity for T_k, T_{k+1} must $= \lambda_k \lambda_{k+1}$. Now from the equations

$$T_k = A_k V - B_k U, \quad T_{k+1} = A_{k+1} V - B_{k+1} U,$$

we have

$$A_{k+1} T_k - A_k T_{k+1} = (A_k B_{k+1} - A_{k+1} B_k) U = \lambda_k \lambda_{k+1} U.$$

Now, comparing the coefficients of the highest powers of x on both sides of the equation, and observing that the highest power does not occur in $A_k T_{k+1}$, we have the product of the leading coefficients of A_{k+1} and $T_k = \lambda_k \lambda_{k+1}$. But if we write

$$\Sigma(a - \beta)^2 = p_2, \ \Sigma(a - \beta)^2 (a - \gamma)^2 (\beta - \gamma)^2 = p_3, \ \&c.,$$

we have, on inspection of the values in Arts. 158, 160, the leading coefficient in $T_2 = p_2$, in $T_3 = p_3$, &c., and in $A_2 = n$, in $A_3 = p_2$, in $A_4 = p_3$, &c. Hence

$$p_2^2 = \lambda_2 \lambda_3, \ p_3^2 = \lambda_3 \lambda_4, \ p_4^2 = \lambda_4 \lambda_5, \ \&c., \text{ whence } \lambda_3 = \frac{p_2^2}{n^2}, \ \lambda_4 = \frac{n^2 p_3^2}{p_2^2} \ \&c.$$

The important matter then is, that these coefficients are all positive squares, and, therefore, as in using Sturm's theorem we are only concerned with the signs of the functions, we may omit them altogether.

162. When we want to know the total number of imaginary roots of an equation, it is well known that we are only con-

cerned with the coefficients of the highest powers of x in Sturm's functions, there being as many pairs of imaginary roots as there are variations in the signs of these leading terms. And since the signs of the leading terms of T_2, T_3, &c. are the same as those of R_2, R_3, &c., it follows that an equation has as many pairs of imaginary roots as there are variations in the series of signs of 1, n, $\Sigma(\alpha-\beta)^2$, $\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2$ &c. This theorem may be stated in a different form by means of Ex. 3, p. 14, and we learn that an equation has as many pairs of imaginary roots as there are variations in the signs of 1, s_0, and the series of determinants

$$\begin{vmatrix} s_0, & s_1 \\ s_1, & s_2 \end{vmatrix}, \quad \begin{vmatrix} s_0, & s_1, & s_2 \\ s_1, & s_2, & s_3 \\ s_2, & s_3, & s_4 \end{vmatrix}, \quad \begin{vmatrix} s_0, & s_1, & s_2, & s_3 \\ s_1, & s_2, & s_3, & s_4 \\ s_2, & s_3, & s_4, & s_5 \\ s_3, & s_4, & s_5, & s_6 \end{vmatrix} \text{ \&c.,}$$

the last in the series being the discriminant; and the condition that the roots of an equation should be all real is simply that every one of these determinants should be positive.*

163. We return now, from this digression on Sturm's theorem to Mr. Borchardt's proof, of which we commenced to give an account, Art. 158; and it is evident that in order to apply the test just obtained, to prove the reality of the roots of the equation got by expanding the determinant of Art. 156, it will be first necessary to form the sums of the powers of the roots of that equation. For the sake of brevity, we confine our proof to the determinant,

$$\begin{vmatrix} a-\lambda, & f, & e \\ f, & b-\lambda, & d \\ e, & d, & c-\lambda \end{vmatrix}$$

it being understood that precisely the same process applies in

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* I do not know whether I have explicitly stated it, but all through these Lessons in expanding a determinant I make the term positive which is obtained by reading from the left-hand top to the right-hand bottom corner, as, for instance, $s_0s_2s_4s_6$ in the last determinant above.

general.* Then it appears immediately, on expanding the determinant, that $s_1 = a + b + c$, since the determinant is of the form $\lambda^3 - \lambda^2 (a + b + c) +$ &c. And in the general case s_1 is equal to the sum of the coefficients of the squares of the variables in the quadric from which the determinant is derived. We can calculate s_2 as follows. The determinant above written is the eliminant of the equations

$$\lambda x = ax + fy + ez, \quad \lambda y = fx + by + dz, \quad \lambda z = ex + dy + cz.$$

Multiply all these equations by λ, and substitute on the right-hand side for λx, λy, λz their values, when we get

$$\begin{aligned}
\lambda^2 x &= (a^2 + f^2 + e^2)x + (af + fb + ed)y + (ae + fd + ec)z, \\
\lambda^2 y &= (fa + bf + de)x + (f^2 + b^2 + d^2)y + (fe + bd + dc)z, \\
\lambda^2 z &= (ea + df + ce)x + (ef + db + cd)y + (e^2 + d^2 + c^2)z,
\end{aligned}$$

from which eliminating x, y, z, we have a determinant of form exactly similar to that which we are discussing, and which may be written

$$\begin{vmatrix} a_2 - \lambda^2, & f_2, & e_2 \\ f_2, & b_2 - \lambda^2, & d_2 \\ e_2, & d_2, & c_2 - \lambda^2 \end{vmatrix};$$

when, of course, in like manner,

$$s_2 = a_2 + b_2 + c_2 = a^2 + b^2 + c^2 + 2d^2 + 2e^2 + 2f^2.$$

The same process applies in general, and enables us from s_p to compute s_{p+1}. Thus, suppose we have got the system of equations

$$\lambda^p x = a_p x + f_p y + e_p z, \; \lambda^p y = f_p x + b_p y + d_p z, \; \lambda^p z = e_p x + d_p y + c_p z.$$

from which we could deduce, as above, $s_p = a_p + b_p + c_p$; then, multiplying both sides by λ, and substituting for λx &c. their values, we shall get

$$\begin{aligned}
\lambda^{p+1} x &= (a_p a + f_p f + e_p e)x + (a_p f + f_p b + e_p d)y + (a_p e + f_p d + e_p c)z \\
\lambda^{p+1} y &= (f_p a + b_p f + d_p e)x + (f_p f + b_p b + d_p d)y + (f_p e + b_p d + d_p c)z \\
\lambda^{p+1} z &= (e_p a + d_p f + c_p e)x + (e_p f + d_p b + c_p d)y + (e_p e + d_p d + c_p c)z
\end{aligned}$$

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* I have, for convenience, changed the sign of λ, which will, of course, not affect the question as to the *reality* of its values.

whence

$$s_{p+1} = a_p a + b_p b + c_p c + 2f_p f + 2e_p e + 2d_p d.$$

164. We shall now show, by the help of these values for s_p &c. and of the principle established Arts. 19, 21, that every one of the determinants at the end of Art. 162 can be expressed as the sum of a number of squares, and is therefore essentially positive. Thus, write down the set of constituents

$$\left\| \begin{matrix} 1, & 1, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ a, & b, & c, & d, & e, & f, & d, & e, & f \end{matrix} \right\|$$

then it is easy to see that $\begin{vmatrix} s_0, & s_1 \\ s_1, & s_2 \end{vmatrix}$ is the determinant formed from this by the method of Arts. 19, 21, and which expresses the sum of all possible squares of determinants which can be formed by taking any two of the nine columns written above. The determinant $\begin{vmatrix} s_0, & s_1 \\ s_1, & s_2 \end{vmatrix}$ is thus seen to be resolvable into the sum of the squares $(a-b)^2 + (b-c)^2 + (c-a)^2 + 6(d^2+e^2+f^2)$, and is therefore essentially positive. Again, if we write down

$$\left\| \begin{matrix} 1, & 1, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \\ a, & b, & c, & d, & e, & f, & d, & e, & f \\ a_2, & b_2, & c_2, & d_2, & e_2, & f_2, & d_2, & e_2, & f_2 \end{matrix} \right\|$$

where a_2 &c. have the meaning already explained, it will be easily seen from the values we have found that $\begin{vmatrix} s_0, & s_1, & s_2 \\ s_1, & s_2, & s_3 \\ s_2, & s_3, & s_4 \end{vmatrix}$ is the determinant which, in like manner, is equal to the sum of the squares of all possible determinants which can be formed out of the above matrix. And so in like manner in general.*

165. We have only to add, in conclusion, that being given

* M. Kummer first found out by actual trial that the discriminant of the cubic which determines the axes of a surface of the second degree is resolvable into a sum of squares. ("Crelle," vol. xxvi., p. 268.) The general theory given here is due, as we have said, to M. Borchardt.

any symmetrical determinant, we can, by an obvious process, write down a quadric which shall have the given determinant for its discriminant. Thus it was proved, p. 35, that the eliminant of any two binary quantics can be expressed as a symmetrical determinant, and Mr. Sylvester forms from that determinant a quadric which he calls the Bezoutiant of the pair of quantics. The discriminant of a quantic can, of course, be expressed in like manner as a symmetric determinant, the quadric formed from which, and which will be a function of $n-1$ variables, may be called the Bezoutiant of the given quantic. The constituents of this determinant can easily be expressed in terms of the roots of the quantic. For when the two quantics, p. 35, are the two differentials of the same quantic, it is seen without difficulty that the determinants (ab'), (ac'), &c. are

$$\Sigma(\alpha-\beta)^2,\quad \Sigma(\alpha-\beta)^2\gamma,\quad \Sigma(\alpha-\beta)^2\gamma\delta,\ \&c.,$$

while the determinants (bc'), (bd'), &c. are

$$\Sigma(\alpha-\beta)^2\gamma^2,\quad \Sigma(\alpha-\beta)^2\gamma^2\delta,\quad \Sigma(\alpha-\beta)^2\gamma^2\delta\epsilon,$$

and so on.

From this symmetric determinant again we can form an equation in λ, as in Art. 156, and Mr. Sylvester has shown that the number of real roots of the given equation is determined by the number of positive roots in this equation in λ. This method of determining the total number of real roots of an equation has a theoretical advantage over Sturm's method, since Sturm's series of functions is essentially unsymmetrical, and we get a different series if, by substituting $\frac{1}{x}$ for x, we write the coefficients of the equation in an inverse order; while this equation in λ is unaffected by such a transformation.*

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* Mr. Sylvester says, p. 513, that the number of real roots in the equation is one more than the number of positive roots in the equation for λ, a statement which is evidently loosely worded, since the equation might not have any real roots at all. But as the coefficients in the equation in λ are not invariants, I have felt less interested in looking for the exact enunciation of Mr. Sylvester's theorem.

LESSON XVI.

HYPERDETERMINANTS—SUPPLEMENTARY LESSON.

166. In order to avoid lengthening Lesson XI. with details which some readers might wish to omit, we have thrown into a supplementary Lesson an account of the methods which the hyperdeterminant calculus affords of expressing one derivative in terms of another. The basis of the whole is the following identical equation. Let D_1 denote $x\dfrac{d}{dx_1} + y\dfrac{d}{dy_1}$, then it is easy to see that we have identically

$$D_1\overline{23} + D_2\overline{31} + D_3\overline{12} = 0, \qquad (A)$$

for, substituting for $\overline{23}$ &c. their values, the coefficients of x and y separately vanish (Art. 3). To illustrate the use to be made of this equation, we write out the first application we make of it in greater fulness of detail than we shall think it necessary to do afterwards. Squaring equation A, which may be written $D_1\overline{23} = D_2\overline{13} - D_3\overline{12}$, we get

$$2D_2D_3\overline{12}.\overline{13} = D_2^2\overline{13}^2 + D_3^2\overline{12}^2 - D_1^2\overline{23}^2. \qquad (B)$$

In this form the equation is true, even if the functions U_1, U_2, U_3, supposed to be operated on, are all different in form. But the case which we shall exclusively consider in this Lesson is when we are forming derivatives of a single function U, with which U_1, U_2, U_3 all become identical when their suffixes are suppressed. And in this case we have seen (Art. 108) that $D_1^2\overline{23}^2$, $D_2^2\overline{31}^2$, $D_3^2\overline{12}^2$ are all only different expressions for the same thing. Hence, equation B becomes

$$D_3^2\overline{12}^2 = 2D_2D_3\overline{12}.\overline{13}.$$

Now, by the theorem of homogeneous functions, the operation D performed on any function only affects it with a numerical multiplier. On the left-hand side of the equation, since D_3^2 only affects U_3 which is not there elsewhere differentiated, that mul-

tiplier is $n(n-1)$; on the right-hand side D_2, D_3, each, affect functions which have been once differentiated besides; and therefore, introduces the numerical multiplier $n-1$; consequently, if we expand the equation $2D_2D_3\overline{12}\cdot\overline{13} = D_3{}^2\overline{12}^2$, drop suffixes, and divide both sides by $2(n-1)$, we get

$$(n-1)\left\{\frac{d^2U}{dx^2}\left(\frac{dU}{dy}\right)^2 - 2\frac{d^2U}{dxdy}\frac{dU}{dx}\frac{dU}{dy} + \frac{d^2U}{dy^2}\left(\frac{dU}{dx}\right)^2\right\}$$
$$= nU\left\{\frac{d^2U}{dx^2}\frac{d^2U}{dy^2} - \left(\frac{d^2U}{dxdy}\right)^2\right\}$$

167. It will be observed, that whenever by transformation the number of figures in the symbol is diminished, it shows that the quantic U is itself a factor in the derivative. Thus, in the example chosen, we proved that $\overline{12}\cdot\overline{13}$ differs only by a numerical factor from $\overline{12}^2$. But as what we operate on is the product $U_1U_2U_3$, and as the symbol $\overline{12}^2$ does not affect U_3 at all; when we drop the suffixes, U remains as a factor.

168. We can show in general that the derivative $\overline{12}^m.\overline{13}$ where m is odd differs only by a numerical multiplier from $\overline{12}^{m+1}$. For multiply equation A by $\overline{12}^m$ and it becomes

$$D_1\overline{12}^m.\overline{23} + D_2\overline{12}^m.\overline{31} + D_3\overline{12}^{m+1} = 0, \text{ or } D_3\overline{12}^{m+1} = 2D_2\overline{12}^m.\overline{13},$$

since the first two terms differ only by the interchange of the figures 1 and 2.

And in general any symbol may be so transformed that the highest power of any factor $\overline{12}$ may be even. For the signification of the symbol is not altered if we interchange the figures 1 and 2; therefore,

$$\overline{12}^{2m+1}\phi_1 = -\overline{12}^{2m+1}\phi_2 = \tfrac{1}{2}\overline{12}^{2m+1}(\phi_1 - \phi_2),$$

and by the help of equation A the quantity $\phi_1 - \phi_2$ can be so transformed as to be divisible by $\overline{12}$. This will be better understood from the annexed example.

Ex. To transform $\overline{12}^3.\overline{13}^2.\overline{14}$ so as to contain $\overline{12}^4$ as a factor—

$$2D_2{}^3\overline{12}^3.\overline{13}^2.\overline{14} = 2D_1{}^3\overline{21}^3.\overline{23}^2.\overline{24} = \overline{12}^3(D_2{}^3\overline{13}^2.\overline{14} - D_1{}^3\overline{23}^2.\overline{24})$$

but if in the last we substitute for $D_1\overline{23}$, $D_2\overline{13} - D_3\overline{12}$, and for $D_1\overline{24}$, $D_2\overline{14} - D_4\overline{12}$,

the quantity within the brackets becomes divisible by $\overline{12}$, and we have

$$2D_2{}^3\overline{12}^3.\overline{13}^2.\overline{14} = \overline{12}^4(D_2{}^2D_4\overline{13}^2 + D_3{}^2D_4\overline{12}^2 + 2D_2{}^2D_3\overline{13}.\overline{14} - D_2D_3{}^2\overline{12}.\overline{14} - 2D_2D_3D_4\overline{12}.\overline{13}).$$

The quantity within the brackets may be further simplified by equation B, when we have

$$4D_2{}^3D_4\overline{12}^3.\overline{13}^2.\overline{14} = \overline{12}^4(6D_2{}^2D_4{}^2\overline{13}^2 - D_3{}^2D_4{}^2\overline{12}^2 - 2D_1{}^2D_2{}^2\overline{34}^2).$$

169. In addition to the identical equation already mentioned, we employ another which can be easily verified, viz.:

$$\overline{12}.\overline{34} - \overline{13}.\overline{24} + \overline{14}.\overline{23} = 0. \qquad (C)$$

By the help of these equations we can reduce all symbols to certain standard forms, which we denote by special letters, as follows. For two factors we take as our standard form the Hessian $\overline{12}^2$ which we call H; for three factors $\overline{12}^2.\overline{13} = G$; for four factors $\overline{12}^4 = I$, or $\overline{12}^2.\overline{34}^2 = H^2$; for five factors $\overline{12}^4.\overline{13} = F$, or $\overline{12}^2.\overline{13}.\overline{45}^2 = GH$; for six factors $\overline{12}^6 = K$, or $\overline{12}^2.\overline{23}^2.\overline{31}^2 = J$, or $\overline{12}^2.\overline{34}^2.\overline{56}^2 = H^3$, and so on. The following examples will sufficiently illustrate how these reductions are effected.

Ex. 1. To express $\overline{12}.\overline{13}.\overline{14}$ in terms of the standard form.

Multiply equation B by $\overline{14}$, when the first two terms on the right-hand side become identical, while the last term vanishes, and we have

$$D_2D_3\overline{12}.\overline{13}.\overline{14} = D_2{}^2\overline{13}^2.\overline{14}, \text{ or } (n-1)\{\overline{12}.\overline{13}.\overline{14}\} = nGU.$$

It is easy to see that $\overline{12}.\overline{13}.\overline{14}$ denotes the result of substituting in a cubic U, $\frac{dU}{dy}$ for x and $-\frac{dU}{dx}$ for y, or, in general, that it denotes the result of a similar substitution in the cubic emanant of any function. While the result of a like substitution in the function itself is $\overline{12}.\overline{13}.\overline{14}.\overline{15}$ &c. Every symbol can, in general, be reduced to a more compact form by substituting for every pair of simple factors having a common figure such as $\overline{12}.\overline{13}$, their value by equation B, and so expressing the given symbol in terms of others in which this pair of factors is replaced by a square factor. In all future examples we shall suppose this reduction to have been made beforehand, and only discuss symbols prepared so as to have as many square factors as possible. Thus, the value of $\overline{12}.\overline{13}.\overline{14}.\overline{15}$ would be calculated by multiplying the equations

$$2D_2D_3\overline{12}.\overline{13} = D_3{}^2\overline{12}^2 + D_2{}^2\overline{13}^2 - D_1{}^2\overline{23}^2;\ 2D_4D_5\overline{14}.\overline{15} = D_5{}^2\overline{14}^2 + D_4{}^2\overline{15}^2 - D_1{}^2\overline{45}^2,$$

when all the terms on the right-hand side of the equation will contain only square factors. The result of the process in any symbol of the form $\overline{12}.\overline{13}.\overline{14}.\overline{15}$ &c. will

be to diminish the number of figures, and, therefore, such a derivative contains U as a factor (Art. 168).

Ex. 2. To calculate $\overline{12}^2.\overline{13}^2$.

Raise to the fourth power $D_1\overline{23} = D_2\overline{13} - D_3\overline{12}$, when, collecting identical terms, we get $6D_2^2D_3^2\overline{12}^2.\overline{13}^2 = 8D_3^3D_2\overline{12}^3.\overline{13} - D_3^4\overline{12}^4$. But, as in Art. 168,

$$8D_3^3D_2\overline{12}^3.\overline{13} = 4D_3^4\overline{12}^4,$$

and it follows that $\overline{12}^2.\overline{13}^2$ also only differs from $\overline{12}^4$ by a numerical multiplier. Or, perhaps, better thus: we know by elementary algebra that the equation $a+b+c=0$ implies $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2=0$. Hence from equation A we have

$$D_1^4\overline{23}^4 + D_2^4\overline{31}^4 + D_3^4\overline{12}^4 - 2D_1^2D_2^2\overline{23}^2.\overline{31}^2 - 2D_2^2D_3^2\overline{31}^2.\overline{12}^2 - 2D_3^2D_1^2\overline{12}^2.\overline{23}^2 = 0, \qquad (D)$$

and

$$2D_2^2D_3^2\overline{12}^2.\overline{13}^2 = D_3^4\overline{12}^4, \text{ or } 2(n-2)(n-3)\{\overline{12}^2.\overline{13}^2\} = n(n-1)IU.$$

Ex. 3. To reduce $\overline{12}^4.\overline{13}^2$. Multiply equation D by $\overline{12}^2$, when we have

$$2D_2^4\overline{13}^4.\overline{12}^2 + D_3^4\overline{12}^6 = 2D_1^2D_2^2\overline{12}^2.\overline{23}^2.\overline{31}^2 + 4D_2^2D_3^2\overline{12}^4.\overline{13}^2;$$

or $2(n-2)(n-3)(n-4)(n-5)\{\overline{12}^4.\overline{13}^2\} = n(n-1)(n-2)(n-3)KU - 2(n-4)^2(n-5)^2J.$

Ex. 4. To reduce $\overline{12}^2.\overline{13}^2.\overline{14}^2$. Multiply the three equations

$$2D_2D_3\overline{12}.\overline{13} = D_3^2\overline{12}^2 + D_2^2\overline{13}^2 - D_1^2\overline{23}^2$$
$$2D_3D_4\overline{13}.\overline{14} = D_4^2\overline{13}^2 + D_3^2\overline{14}^2 - D_1^2\overline{34}^2$$
$$2D_4D_2\overline{14}.\overline{12} = D_2^2\overline{14}^2 + D_4^2\overline{12}^2 - D_1^2\overline{24}^2$$

when, after assembling the identical terms, and reducing, we have

$$6D_2^2D_3^2D_4^2\overline{12}^2.\overline{13}^2.\overline{14}^2 = -4D_4^6\overline{12}^2.\overline{23}^2.\overline{31}^2 - 3D_1^2D_2^2D_3^2\overline{14}^4.\overline{23}^2 + 6D_3^2D_4^4\overline{12}^4.\overline{13}^2,$$

where, as the value of $\overline{12}^4.\overline{13}^2$ has been already calculated, the value of $\overline{12}^2.\overline{13}^2.\overline{14}^2$ is determined.

Ex. 5. To reduce $\overline{12}^2.\overline{13}^2.\overline{34}^2$. We may either by multiplying equation D by $\overline{34}^2$ obtain an expression for the derivative in question in terms of $\overline{12}^4.\overline{13}^2$, and $\overline{12}^2.\overline{13}^2.\overline{14}^2$ which have been already calculated, or else proceed directly as follows. Multiply by $\overline{14}^2$ the product of

$2D_2D_3\overline{12}.\overline{13} = D_3^2\overline{12}^2 + D_2^2\overline{13}^2 - D_1^2\overline{23}^2$; $2D_2D_3\overline{24}.\overline{34} = D_3^2\overline{24}^2 + D_2^2\overline{34}^2 - D_4^2\overline{23}^2$, and we have

$$4D_2^2D_3^2\overline{12}.\overline{13}.\overline{24}.\overline{34}.\overline{14}^2 = 2D_3^4\overline{12}^2.\overline{24}^2.\overline{41} + D_1^2D_4^2\overline{23}^4.\overline{14}^2 - 2D_2^2D_4^2\overline{14}^2.\overline{13}^2.\overline{23}^2$$

But from equation C

$$2.\overline{14}^2(\overline{12}.\overline{13}.\overline{24}.\overline{34}) = \overline{14}^2(\overline{12}^2.\overline{34}^2 + \overline{13}^2.\overline{24}^2 - \overline{14}^2.\overline{23}^2),$$

therefore, the left-hand side of the preceding equation becomes

$$4D_2^2D_3^2\overline{12}^2.\overline{14}^2.\overline{34}^2 - 2D_2^2D_3^2\overline{14}^4.\overline{23}^2,$$

and hence reducing, we have

$$6D_2^2D_3^2\overline{12}^2.\overline{14}^2.\overline{34}^2 = 3D_2^2D_3^2\overline{14}^4.\overline{23}^2 + 2D_3^4\overline{12}^2.\overline{24}^2.\overline{41}^2,$$

or $$6(n-2)(n-3)\overline{12}^2.\overline{14}^2.\overline{34}^2 = 3(n-2)(n-3)HI + 2n(n-1)JU.$$

170. We wish next to show how to form the symbol expressing any derivative of a derivative. In the first place it is obvious that if we have any function of x_1, x_2, x_3, &c., we shall get the same result whether we suppress all the suffixes and then differentiate with regard to x, or whether we take the sum of the differentials with regard to x_1, x_2, x_3, &c., and then suppress the suffixes. The differential, then, with regard to x of any derivative symbol containing the figures 1, 2, 3, &c. (such as $\overline{12}^2.\overline{13}^2$) is got by operating on this symbol with $\frac{d}{dx_1}+\frac{d}{dx_2}+\frac{d}{dx_3}+$ &c. The manner in which other derivatives are expressed will sufficiently appear if we take a particular example, and show how to express symbolically the Hessian of the Hessian. It will be remembered that the Hessian itself is expressed by forming the product of U_1 by the same function written with different letters U_2, and then operating with $(\zeta_1\eta_2-\xi_2\eta_1)^2$ where ξ, η denote differentials. To form the Hessian of $\overline{12}^2$ we form the product $\overline{12}^2.\overline{34}^2$ and operate on it with $(\xi_1\eta_2-\xi_2\eta_1)^2$ where $\xi_1=\frac{d}{dx_1}+\frac{d}{dx_2}$, $\xi_2=\frac{d}{dx_3}+\frac{d}{dx_4}$ &c. And thus it is easy to see that the Hessian of the Hessian is $(\overline{13}+\overline{14}+\overline{23}+\overline{24})^2\overline{12}^2.\overline{34}^2$, which expanded is equivalent to

$$4\,(\overline{12}^2.\overline{13}^2.\overline{34}^2)+4\,(\overline{12}^2.\overline{34}^2.\overline{13}.\overline{24})+8\,(\overline{13}.\overline{14}.\overline{12}^2.\overline{34}^2).$$

Or, to take a more complicated example, let us endeavour to express symbolically the result of performing the operation $\overline{12}^2.\overline{23}^2.\overline{31}^2$ on three different functions, the first being the Hessian $\overline{12}^2$, the second being J, or $\overline{12}^2.\overline{23}^2.\overline{31}^2$, and the third the function U itself. Then we must use different figures for the two derivatives, and we therefore form the product of $\overline{12}^2$ by $34^2.45^2.53^2$, and operate on it with

$$(\xi_1\eta_2-\xi_2\eta_1)^2\,(\xi_2\eta_3-\xi_3\eta_2)^2\,(\xi_3\eta_1-\xi_1\eta_3)^2,$$

where we are to put for ξ_1, $\frac{d}{dx_1}+\frac{d}{dx_2}$; for ξ_2, $\frac{d}{dx_3}+\frac{d}{dx_4}+\frac{d}{dx_5}$; and for ξ_3, $\frac{d}{dx_6}$; and the symbol required is found by expanding

$$(\overline{13}+\overline{14}+\overline{15}+\overline{23}+\overline{24}+\overline{25})^2(\overline{36}+\overline{46}+\overline{56})^2$$
$$(\overline{16}+\overline{26})^2\,\overline{12}^2.\overline{34}^2.\overline{45}^2.\overline{53}^2.$$

171. The symbolical expression just given for the Hessian of the Hessian enables us to prove that it can be reduced to the form $IH+\lambda JU$, as we saw otherwise, p. 104. In fact, we have already seen, Ex. 5, p. 139, that $\overline{12}^2.\overline{13}^2.\overline{34}^2$ can be expressed in the required form, so that it remains only to prove the same thing for the other terms in the expression given in the last Article for the Hessian of the Hessian.

Multiply by $\overline{12}^2$ the product of the equations

$$-2D_4D_1\overline{13}.\overline{34}=D_1^2\overline{34}^2+D_4^2\overline{13}^2-D_3^2\overline{14}^2;$$
$$2D_2D_3\overline{24}.\overline{34}=D_2^2\overline{34}^2+D_3^2\overline{24}^2-D_4^2\overline{23}^2,$$

when we have

$$-4D_1D_2D_3D_4\overline{12}^2.\overline{34}^2.\overline{13}.\overline{24}=D_1^2D_2^2\overline{12}^2.\overline{34}^4$$
$$+2D_3^2D_4^2\overline{12}^2.\overline{13}^2.\overline{24}^2-2D_4^4\overline{12}^2.\overline{23}^2.\overline{31}^2,$$

which, putting in for $\overline{12}^2.\overline{13}^2.\overline{24}^2$, its value already found, gives

$$-12(n-3)^3\{\overline{12}^2.\overline{34}^2.\overline{13}.\overline{24}\}=6(n-2)^2(n-3)IH$$
$$-4n(n-1)(n-2)JU.$$

Again, to calculate $\overline{12}^2.\overline{34}^2.\overline{13}.\overline{14}$, we have only to substitute for $\overline{13}.\overline{14}$ from equation B, when we have

$$2D_3D_4\{\overline{12}^2.\overline{34}^2.\overline{13}.\overline{14}\}=2D_4^2\overline{12}^2.\overline{34}^2.\overline{13}^2-D_1^2\overline{12}^2.\overline{34}^4,$$

and substituting as before, we get

$$6(n-3)^2\{\overline{12}^2.\overline{34}^2.\overline{13}.\overline{14}\}=2n(n-1)JU,$$

by the help of which values the Hessian of the Hessian is expressed in the desired form.

172. We proceed now to give a few examples of the application of the same method to ternary quantics. The identical equations which we now principally use are

$$D_4\overline{123}=D_1\overline{234}-D_2\overline{134}+D_3\overline{124} \qquad (E)$$
$$\overline{123}.\overline{145}+\overline{124}.\overline{153}+\overline{125}.\overline{134}=0 \qquad (F)$$

to which may be added the corresponding equations for contravariant symbols

$$P\overline{123} = D_1\overline{a23} + D_2\overline{a31} + D_3\overline{a12} \qquad (G)$$

$$\overline{a12}.\overline{a34} + \overline{a23}.\overline{a14} + \overline{a31}.\overline{a24} = 0 \qquad (H)$$

where P is $\alpha x + \beta y + \gamma z$.

We have already given the symbolical expression

$$\overline{123}.\overline{124}.\overline{234}.\overline{314}$$

for the S of a cubic; and the principles explained (Art. 170) enable us readily to obtain the symbolical expression of T, the latter invariant being defined as the result of operating on the Hessian with the evectant of S (see Art. 145). For we have only to substitute $4 + 5 + 6$ for a in the evectant symbol, viz.,

$$\overline{123}.\overline{a12}.\overline{a23}.\overline{a31},$$

and operate on $\overline{456}^2$, when, on expanding the symbol and rejecting all the terms in which the differentials exceed the third degree, we have remaining only terms of the form

$$\overline{123}.\overline{412}.\overline{523}.\overline{631}.\overline{456}^2,$$

which is therefore an expression for T. We add some examples of the application of the identical equations (E), (F), (G), (H).

Ex. 1. We commence by proving the relation which, as we asserted, p. 122, exists between the quartic covariants of a quartic. Let α, β, γ, δ be any quantities connected by the relation $\alpha + \beta + \gamma + \delta = 0$, then, by twice squaring, it is easy to see that $8\alpha\beta\gamma\delta = 2\Sigma\alpha^2\beta^2 - \Sigma\alpha^4$. Now, applying this principle to equation (E), we get

$$8D_1D_2D_3D_4\overline{123}.\overline{124}.\overline{234}.\overline{314} = 4D_4^4\overline{123}^4 - 12D_3^2D_4^2\overline{123}^2.\overline{124}^2$$

which is the relation required.

Ex. 2. Let us take as our next example to prove that Φ at the foot of p. 114 reduces, as we there saw, to US in the case of a cubic.

We are to substitute in $\overline{a12}^2$, $4 + 5 + 6$ for a, and operate on $\overline{456}^2$. Omitting the terms in which any figure exceeds the third degree, the symbol reduces to $\overline{124}.\overline{125}.\overline{456}^2$ which we may write $\overline{123}^2.\overline{145}.\overline{245}$. But, multiplying the two identities

$$\overline{123}.\overline{145} = \overline{124}.\overline{135} - \overline{125}.\overline{134};\quad \overline{123}.\overline{245} = \overline{124}.\overline{235} - \overline{125}.\overline{234}$$

we get

$$\overline{123}^2.\overline{145}.\overline{245} = 2\,(\overline{124}.\overline{125}.\overline{134}.\overline{235}).$$

Again, multiplying by $\overline{124}\,.\,\overline{134}\,.\,\overline{235}$ the equation

$$D_5\overline{123} = D_1\overline{235} + D_2\overline{315} + D_3\overline{125},$$

we get

$$D_5\overline{123}\,.\,\overline{124}\,.\,\overline{134}\,.\,\overline{235} = D_1\overline{124}\,.\,\overline{134}\,.\,\overline{235}^2 + 2D_3\overline{124}\,.\,\overline{125}\,.\,\overline{134}\,.\,\overline{235},$$

or, since we have proved the two terms on the right-hand side to be identical,

$$(n-1)\overline{123}\,.\,\overline{124}\,.\,\overline{134}\,.\,\overline{235} = 2\,(n-2)\overline{123}^2\,.\,\overline{145}\cdot\overline{245},$$

while finally multiplying by $\overline{123}\cdot\overline{124}\cdot\overline{134}$ the equation

$$D_5\overline{234} = D_2\overline{345} + D_3\overline{425} + D_4\overline{235},$$

we get

$$D_5\overline{123}\cdot\overline{124}\cdot\overline{234}\cdot\overline{134} = 3D_4\overline{123}\cdot\overline{124}\cdot\overline{134}\cdot\overline{235},$$

or

$$nSU = 3\,(n-2)\,\overline{123}\cdot\overline{124}\cdot\overline{134}\cdot\overline{235},$$

whence

$$6\,(n-2)^2\overline{123}^2\cdot\overline{145}\cdot\overline{245} = n\,(n-1)SU.$$

Ex. 3. In the theory of double tangents to plane curves, explained, "Higher Plane Curves," p. 81,* it is necessary to calculate the result of substituting in the successive emanants $\gamma\dfrac{dU}{dy} - \beta\dfrac{dU}{dz}$, $\alpha\dfrac{dU}{dz} - \gamma\dfrac{dU}{dx}$, $\beta\dfrac{dU}{dx} - \alpha\dfrac{dU}{dy}$, for x, y, z; and to show that each result is of the form $P_nU + Q_n(\alpha x + \beta y + \gamma z)^2$. We shall in this and in the next two examples perform the calculation of Q_2, Q_3, Q_4. The symbolical expression for the result of substitution, it is easy to see, is $\overline{a12}\cdot\overline{a13}\cdot\overline{a14}\cdot\overline{a15}$ &c.

Now first to calculate $\overline{a12}\cdot\overline{a13}$, we have only to square equation (G), when we get

$$P^2\overline{123}^2 = 3D_3{}^2\overline{a12}^2 - 6D_2D_3\overline{a12}\cdot\overline{a13}.$$

or if we denote the Hessian $\overline{123}^2$ by H, and the bordered Hessian $\overline{a12}^2$ by G, we have

$$6\,(n-1)^2\overline{a12}\,.\,\overline{a13} = 3n\,(n-1)\,GU - P^2H.$$

Ex. 4. To calculate $\overline{a12}\cdot\overline{a13}\cdot\overline{a14}$. From equation ($G$) we have

$$-2D_2D_3\overline{a12}\cdot\overline{a13} = P^2\overline{123}^2 - 2PD_1\overline{123}\cdot\overline{a23} + D_1{}^2\overline{a23}^2 - D_2{}^2\overline{a31}^2 - D_3{}^2\overline{a12}^2.$$

Multiply by $\overline{a14}$, two of the terms vanish identically, and

$$-2\,(n-1)^2\overline{a12}\cdot\overline{a13} = P^2\overline{123}^2\cdot\overline{a14} - 2n\,(n-1)\,U\overline{a12}^2\cdot\overline{a13}.$$

To prove that $\overline{123}\cdot\overline{a23}\cdot\overline{a14}$ vanishes identically, we have only to multiply equation (H) by $\overline{123}$, when since the terms in it differ only by a permutation of the figures 1, 2, 3, each must separately = 0. In like manner, it is proved that $\overline{123}\cdot\overline{a23}\,.\,\overline{a14}\,.\,\overline{a15}$ is identically = 0, of which use is made in the next example.

Ex. 5. To calculate $\overline{a12}\cdot\overline{a13}\cdot\overline{a14}\cdot\overline{a15}$. We must multiply together the equations

* I have recently published a different method of obtaining the points of contact of double tangents: see "Philosophical Magazine," October, 1858.

$$2D_2D_3\overline{a12}\cdot\overline{a13} = P^2\overline{123}^2 - 2PD_1\overline{123}\cdot\overline{a23} + D_1^2\overline{a23}^2 - D_2^2\overline{a31}^2 - D_3^2\overline{a12}^2$$
$$-2D_4D_5\overline{a14}\cdot\overline{a15} = P^2\overline{145}^2 - 2PD_1\overline{145}\cdot\overline{a45} + D_1^2\overline{a45}^2 - D_4^2\overline{a51}^2 - D_5^2\overline{a14}^2.$$

The product, we know, must be of the form $P_4U + Q_4P^2$, and for brevity we only investigate the value of Q_4, and omit that part which is multiplied by U, indicated by one or more figures disappearing from the symbolical expression. The result then will be

$$4(n-1)^4(\overline{a12}\cdot\overline{a13}\cdot\overline{a14}\cdot\overline{a15}) = P^4\overline{123}^2.\overline{145}^2 - 4P^3D_1\overline{123}^2.\overline{145}\cdot\overline{a45}$$
$$+ 2P^2D_1^2\overline{123}^2.\overline{a45}^2 + 4P^2D_1^2\overline{123}\cdot\overline{a23}\cdot\overline{145}\cdot\overline{a45} - 4PD_1^3\overline{123}\cdot\overline{a23}\cdot\overline{a45}^2.$$

But the last two terms destroy each other, as appears on substituting in the first of them for $P\overline{145}$, $D_1\overline{a45} + D_4\overline{a51} + D_5\overline{a14}$, when the difference vanishes identically. It remains, therefore, that

$$4(n-1)^4Q_4 = P^2\overline{123}^2.\overline{145}^2 - 4(n-3)P\overline{123}^2.\overline{145}\cdot\overline{a45} + 2(n-2)(n-3)HG.$$

where H is the Hessian, and G the bordered Hessian.

Ex. 6. To determine the double tangent curve of a quartic it is necessary ("Higher Plane Curves," p. 88) to calculate $Q_3^2 = 3Q_2Q_4$ where Q_2, Q_3, Q_4, have the values found in Ex. 3, 4, 5, that is to say $6(n-1)^2Q_2 = -H$, and

$$-2(n-1)^2Q_3 = \overline{123}^2.\overline{a14}.$$

We have hence

$$4(n-1)^4Q_3^2 = \overline{123}^2\cdot\overline{a17}\cdot\overline{456}^2\cdot\overline{a48}.$$

But

$$D_8\overline{a17} - D_7\overline{a18} = P\overline{178} - D_1\overline{a78}$$
$$D_8\overline{a47} - D_7\overline{a48} = P\overline{478} - D_4\overline{a78}.$$

Multiplying, we have

$$-2D_7D_8\overline{a17}\cdot\overline{a48} + 2D_8^2\overline{a17}\cdot\overline{a47} = P^2\overline{178}\cdot\overline{478} - 2PD_4\overline{178}\cdot\overline{a78} + D_1D_4\overline{a78}^2.$$

Multiplying by $123^2.456^2$ and omitting a term multiplied by U, we have

$$-8(n-1)^6Q_3^2 = P^2\overline{123}^2.\overline{456}^2.\overline{178}\cdot\overline{478} - 2(n-2)PH\overline{123}^2.\overline{178}\cdot\overline{a78} + (n-2)^2H^2G.$$

And from the values already found, for Q_2 and Q_4

$$2(n-3)Q_3^2 - 3(n-2)Q_2Q_4$$

is proportional to

$$2(n-3)\overline{123}^2.\overline{456}^2.\overline{178}\cdot\overline{478} - (n-2)H\overline{123}^2.\overline{145}^2.$$

The first term may easily be seen (see Art. 170) to be the symbolical expression for Θ, which is the result of substituting $\frac{dH}{dx}$ &c. for x &c. in the bordered Hessian. The covariant $\overline{123}^2.\overline{145}^2$ may be easily calculated by squaring the bordered Hessian, and then substituting $\frac{d}{dx}$ &c. for a &c. This covariant only differs from that called Φ (Art. 145) by terms multiplied by U.

NOTE ON COMMUTANTS.

Having for the sake of brevity omitted all mention of commutants in the foregoing Lessons, we add a note to explain the meaning of the word. It will be more easily understood if we first explain what Mr. Sylvester has called his *umbral* notation for determinants. Consider, for example, the determinant

$$\begin{vmatrix} a\alpha, & b\alpha, & c\alpha, & d\alpha \\ a\beta, & b\beta, & c\beta, & d\beta \\ a\gamma, & b\gamma, & c\gamma, & d\gamma \\ a\delta, & b\delta, & c\delta, & d\delta \end{vmatrix}$$

the constituents of which are $a\alpha$, $b\alpha$, &c., and where a, b, c, &c. are not quantities, but, as it were, shadows of quantities; that is to say, they have no meaning separately, except in combination with one of the other class of umbræ α, β, γ, δ. Thus, for example, if α, β, γ, δ represent the suffixes 1, 2, 3, 4, the constituents in the notation we have used p. 8, &c., are all formed by combining one of the letters a, b, c, d with one of the figures 1, 2, 3, 4. Now the determinant above written may be written more compactly

$$\begin{matrix} a, & b, & c, & d, \\ \alpha, & \beta, & \gamma, & \delta, \end{matrix}$$

which denotes the sum of all possible products of the form $a\alpha . b\beta . c\gamma . d\delta$ obtained by giving the terms in the second line every possible permutation, and changing sign according to the ordinary rule with every permutation.

So again, let us for brevity write ξ, η for $\frac{d}{dx}$, $\frac{d}{dy}$, then it is easy to see what according to the same rule would be meant by

$$\begin{matrix} \xi, & \eta, & & \xi^2, & \xi\eta, & \eta^2. \\ \xi, & \eta, & & \xi^2, & \xi\eta, & \eta^2. \end{matrix}$$

We compound the partial constituents in each column in order to find the factors in the product we want to form, and we take the sum with proper signs of all possible products obtained by permuting the terms

in the lower row. Thus the first example denotes $\xi^2.\eta^2 - \xi\eta.\xi\eta$, which is the Hessian; and the second denotes $\xi^4.\xi^2\eta^2.\eta^4 - \xi^4.\xi\eta^3.\xi\eta^3$ &c., which is the ordinary cub-invariant of a quartic.

Again, since multiplication is performed by addition of indices, it will be readily understood that we can equally form commutants where the partial constituents are combined by addition instead of by multiplication. Thus, considering the quantics

$$(a_2,\ a_1,\ a_0 \between x,\ y)^2,\ (a_4,\ a_3,\ a_2,\ a_1,\ a_0 \between x,\ y)^4,$$

the invariants in the last two examples may be written

$$\begin{matrix} 1, & 0 \\ 1, & 0 \end{matrix} \qquad\qquad \begin{matrix} 2, & 1, & 0 \\ 2, & 1, & 0 \end{matrix}$$

which expanded are $a_2a_0 - a_1a_1$; $a_4a_2a_0 - a_4a_1a_1$ + &c.

All these commutants with only two rows may be written as determinants, but it is a natural extension of the above notation to form commutants with more than two rows, such as

$$\begin{matrix} \xi, & \eta \\ \xi, & \eta \\ \xi, & \eta \\ \xi, & \eta \end{matrix} \qquad\qquad \begin{matrix} 1, & 0 \\ 1, & 0 \\ 1, & 0 \\ 1, & 0 \end{matrix} \qquad\qquad \begin{matrix} \xi^2, & \xi\eta, & \eta^2. \\ \xi^2, & \xi\eta, & \eta^2. \\ \xi^2, & \xi\eta, & \eta^2. \\ \xi^2, & \xi\eta, & \eta^2. \end{matrix}$$

These all denote the sum of a number of products, each product consisting of as many factors as there are columns in the commutant, and each factor being formed by compounding the constituents of the same column; and where we permute in every possible way the constituents in each row after the first. Thus the first and second examples denote the same thing, namely, the quadr-invariant of a quartic expressed in either of the forms $\xi^4.\eta^4 - 4\xi^3\eta.\xi\eta^3 + 3\xi^2\eta^2.\xi^2\eta^2$ or $a_4a_0 - 4a_3a_1 + 3a_2a_2$, while the third example $\xi^8.\xi^4\eta^4.\eta^8$ – &c. denotes the cub-invariant of an octavic given at length, p. 81.

So, in like manner, the invariant expanded, p. 117, may be written as a commutant

$$\begin{matrix} x, & y, & z \\ x, & y, & z \\ \alpha, & \beta, & \gamma \\ \alpha, & \beta, & \gamma. \end{matrix}$$

We have seen that the two invariants of a binary quartic can be expressed as commutants, but it will be found impossible to express in

the same way the discriminant of a cubic. Thus, the leading term in it being $a_3^2 a_0^2$ or $\xi_3 \xi_3 \eta_3 \eta_3$, we are naturally led to expect that it might be the commutant

$$\begin{matrix} \xi, & \eta, & \xi, & \eta, \\ \xi, & \eta, & \xi, & \eta, \\ \xi, & \eta, & \xi, & \eta, \end{matrix}$$

but this commutant, instead of giving the discriminant, will be found to vanish identically. It may, however, be made to yield the discriminant by placing certain restrictions on the permutations which are allowable. For further details I refer to the papers of Messrs. Cayley and Sylvester in the "Cambridge and Dublin Mathematical Journal," 1852.

THE END.